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A SIMPLE METHOD FOR TOMOGRAPHY RECONSTRUCTION BASED ON A DISCRETE VERSION OF THE TOPOLOGICAL GRADIENT

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ABSTRACT

The problem of reconstructing an image from projections has been largely studied. Nowadays, the so called "Direct methods" (based on Fourier transform) are the most widely used, mainly because of its low computational cost. Nevertheless, in some situations old fashion iterative algorithms are more appropriate as is the case of Nuclear imaging, where attenuation coefficients need to be considered in the reconstruction process. In this work we present a reconstruction method based on the well established concept of topological sensitivity analysis for a simple tomography model. This concept is based on analyzing the sensitivity of a cost function to small perturbations in the domain topology or material properties. In particular, the tomography model is used to reconstruct the attenuation coefficient from 1D projections thought 2D slices. After revisiting the topological expansion concept and presenting the cost function under consideration, the topological gradient is computed. In the results section the topological gradient information is used to devise an iterative reconstruction algorithm and finally, some results are shown.

INTRODUCTION

The inverse problem associated to the reconstruction of the 3D data based on 2D projections has been largely studied. Different alternatives have been proposed over the years [11, 15]. Early approaches for image reconstruction are based on iterative processes. Although these methods were the most popular in the early days of Computed Tomography (CT), they present a high computational cost and its convergence accuracy is compromised by the presence of noise. For these reasons, they became almost completely replaced by direct methods. Nowadays, CT reconstruction is driven by direct methods based on Fourier Transform (i.e., Filtered Back Projection - FBP), mainly by the significant computational time reduction.

Nevertheless, situations exist where an iterative algorithm is more convenient than a direct one. For example, in the case of Nuclear Imaging (Single Photon Emission Tomography - SPECT and Positron Emission Tomography - PET), where an internal source is used, direct algorithms do not take into account the absorption of the tissues that surround the illuminating source. This characteristic, produces an attenuation on the resulting image that may lead to a miss judgement by the specialist. On the other hand, iterative methods allow the introduction of attenuation correction what may lead to more precise reconstructions.

The objective of this work is to present an alternative reconstruction method based on the well established concept of topological sensitivity analysis [14, 17, 18]. This concept is based on analyzing the sensitivity of a specific cost function to a change in the domain topology or material properties. In particular, we use a simple tomography model to represent the attenuation of 1D projections thought 2D slices (extending these ideas to 3D is straightforward). This work is organized as follows: First the simplified tomography model is described and the tomography reconstruction problem is stated. After that, the topological gradient concept is revisited together with the addressed cost function and the topological gradient calculation. This gradient is calculated taking the misfit between a measurement and the model solution as cost function. In the numerical results section an iterative algorithm that uses this gradient is introduced and some results are also shown.

A SIMPLE TOMOGRAPHY MODEL

When an object (e.g., a body or tissue) is irradiated with an X-ray source, the incident ray is attenuated by two different phenomena: absorption and dispersion. For simplicity, no distinction is made between both effects (considering them together). In order to present the idea, let us consider an object composed by an homogeneous material. If we assume that the object is being illuminated by n rays from the side of the object where the source is located, after an arbitrary period of time during which the object was irradiated, only $n + \triangle n$ rays where able to pass though the object (assuming $\triangle n$ is negative, where every ray is assumed to have the same energy). In this situation, the following relation is satisfied

$$\frac{\triangle n}{n}\frac{1}{\triangle s} = -\mu,$$

being μ the rate of photons loss due to both phenomena (which we will call attenuation) and s the variable used to parameterize the path line passed by the ray. Taking $\triangle s \rightarrow 0$ we obtain the following differential equation

$$\frac{1}{n}dn = -\mu ds.$$

To find the solution of this equation we integrate along the path of the ray through the object using the model presented in Fig. 1, considering μ not constant and depending on the spatial variable $\mathbf{x}$ (i.e., $\mu(\mathbf{x})$, that can be parameterized using s as $\mathbf{x}(s)$). Assuming the thickness of the ray is small enough, we obtain

$$\begin{aligned} n_{out} &= n_{in} \exp\left[-\int_r \mu(s)ds\right] \\ &\Rightarrow \int_r \mu(s)ds = \ln\frac{n_{in}}{n_{out}}, \end{aligned}$$

being n_{out} the number of photons reaching the sensor for a non homogeneous object, ds is a differential element along the path line r of the ray. The left-hand-side corresponds to a ray integral of a projection. Therefore, measurements like $\ln\frac{n_{in}}{n_{out}}$ taken from different angles, may be used to generate projections of data $\mu(s)$. That is

$$\ln n_{in} - \int_r \mu(s)ds = \ln n_{out}.$$

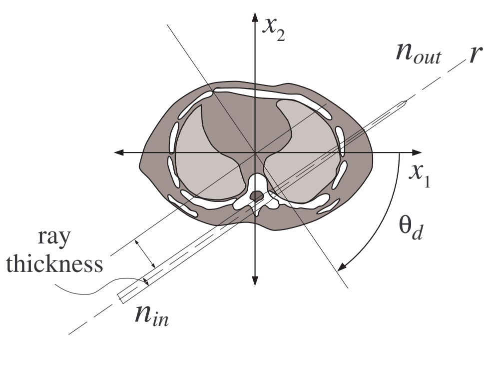


Figure 1: Ray attenuation path.

If we consider that the object is being illuminated by m rays in each direction, the expression presented above can be rewritten as

$$\alpha_j - \int_{r_j} \mu(s)ds = p_j, \ \ j = 1, 2, \ldots, m$$

being α the intensity of the ray illuminating the object and p the measurement observed at the sensor on the other side of the object. If the object is illuminated in D different directions, $d = 1\ldots D$ different projections of μ are obtained, namely

$$\alpha_j^d - \int_{r_j^d} \mu(s)ds = p_j^d, \tag{1}$$

In Eq. (1) the tomography model is assumed to be continuous. In order to solve this problem computationally this continuous model needs to be replaced by a discretized one. To do this, the continuous representation of the attenuation coefficient $\mu(\mathbf{x})$ (Fig. 2(a)) is divided using a regular grid (Fig. 2(b)). The values of $\mu(\mathbf{x})$ are assumed to be constant on each cell of the grid [19]. Then, let μ_i be the constant value of the i^{th} cell, and N the total number of cells.

In this case, the ray is assumed to be a thick line running trough the plane spanned by two linearly independent vectors x_1 and x_2. The integral in Eq. (1) can be substituted by a ray-sum p_j measured for the j^{th} ray. We may express the relationship between μ_i and p_j as following

$$\alpha_j - \sum_{i=1}^{N} K_{ij}\mu_i = p_j, \ \ j = 1, 2, \ldots, M \tag{2}$$

where $M = m * D$ is the total number of rays in all the projections, and K_{ij} is a weighting factor that represents the contribution of the i^{th} cell to the j^{th} ray integral (i.e., the area of intersection between the ray and the cell). For simplicity, let us assume the following notation

$$\mathbf{p} = (p_1, p_2, \ldots, p_j, \ldots, p_M)^T,$$

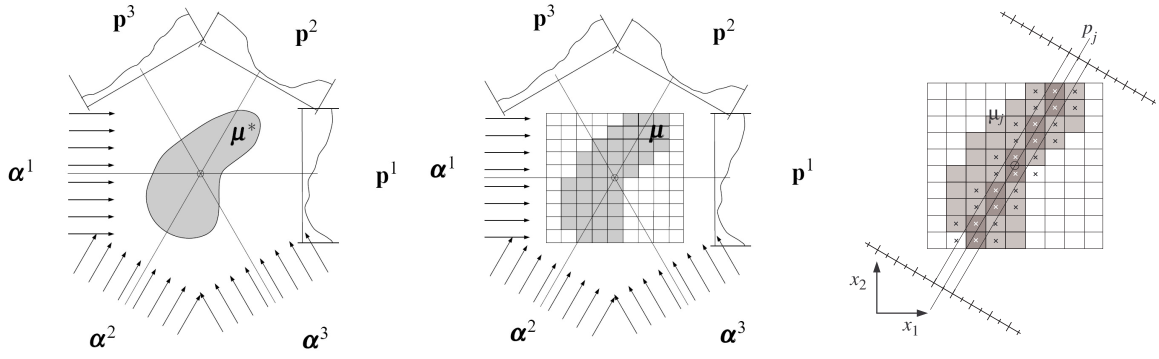
$$\mathbf{K} = \begin{pmatrix} K_{11} & K_{12} & \ldots & K_{1N} \\ K_{21} & K_{22} & & \\ \vdots & & \ddots & \\ K_{M1} & & & K_{MN} \end{pmatrix},$$

$$\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_i, \ldots, \mu_M)^T.$$

Therefore, Eq. (2) can be rewritten in compact form as following

$$\mathbf{p} = \boldsymbol{\alpha} - \mathbf{K}\boldsymbol{\mu}.$$

Note that only a small number of K_{ij}'s are different from zero since only a small number of cells is passed through by any given ray.



(a) Real obstacle. (b) Approximate model. (c) Projection example.

Figure 2: Simplified tomography model.

A TOMOGRAPHY RECONSTRUCTION PROBLEM

The tomography reconstruction problem under consideration can be stated as: *given the measurement* $\mathbf{p}^*$*, find an approximation* $\boldsymbol{\mu}$ *of the unknown attenuation coefficient* $\boldsymbol{\mu}^*$ *by solving the following equation*

$$\mathbf{K}\boldsymbol{\mu} = \boldsymbol{\alpha} - \mathbf{p}^*. \tag{3}$$

For large values of M and N different iterative methods exist for solving Eq. (3). For example, the so called "method of projections", first proposed in [10] and reviewed in [21]. The computational procedure for finding the solution consists in starting with an initial guess $\boldsymbol{\mu}^0$ and successively projecting it on the subspaces defined by the rows of $\mathbf{K}$. If a unique solution of the system exists, the iterations will converge to that solution.

Different algorithms based on these ideas have been proposed over the years (ART - Algebraic Reconstruction Technique [7], SART - Simultaneous Algebraic Reconstruction Technique [2], etc.).

As mentioned before, in this work we propose an alternative tomography reconstruction method based on the topological gradient. This gradient allows us to quantify the sensitivity of a problem when the domain under consideration is perturbed by changing its topology, for example by the introduction of an arbitrary shaped hole, an inclusion or a source term. Early work on this subject can be found in papers by Masmoudi, Sokolowsky and their co-workers [13, 20]. This derivative has been originally conceived as a tool to solve topology optimization problems. Nevertheless, this concept is wider and has shown interesting results when applied in inverse problems. See also [1, 5, 6, 14, 16, 17] for applications of the topological derivative to the above problems considering Navier, Laplace, Poisson, Helmoltz, Stokes and Navier-Stokes equations among others. Most lately, this concept has also been applied in image processing [3, 4, 9, 8, 12].

In particular, we study the behavior of a properly defined cost function when the attenuation coefficients of the object being illuminated is perturbed. With this information, an iterative algorithm is proposed.

TOPOLOGICAL DERIVATIVE CONCEPT

Let us consider a cost function $\Psi(\boldsymbol{\mu})$. If we perturb $\boldsymbol{\mu}$ (say, $\boldsymbol{\mu}_T = \boldsymbol{\mu} + \delta\boldsymbol{\mu}$) we obtain a perturbed cost function $\Psi(\boldsymbol{\mu}_T)$, then for small perturbations $\delta\boldsymbol{\mu}$, the following expansion holds

$$\Psi(\boldsymbol{\mu}_T) = \Psi(\boldsymbol{\mu}) + \mathbf{g}\Psi \cdot \delta\boldsymbol{\mu} + O(\delta\boldsymbol{\mu}), \tag{4}$$

where $\delta\boldsymbol{\mu}$ is such that

$$\delta\boldsymbol{\mu} = \boldsymbol{\mu}_T - \boldsymbol{\mu}$$

and the perturbed attenuation coefficient is

$$\boldsymbol{\mu}_T \quad = \quad (\mu_1, \mu_2, \ldots, \mu_T, \ldots, \mu_M)^T,$$

that is, the value of $\boldsymbol{\mu}$ at cell i was changed from μ_i to μ_T, then

$$\delta\boldsymbol{\mu} \quad = \quad (0, 0, \ldots, \mu_T - \mu_i, \ldots, 0)^T,$$

meaning that the perturbation is made in one cell. Therefore, $\mathbf{g}\Psi$ can be recognized as a discrete version of the topological gradient.

In order to minimize the cost function, $\mathbf{g}\Psi \cdot \delta\boldsymbol{\mu}$ should always be negative. Thus, according to the sign of the topological gradient $\mathbf{g}\Psi$, we need to choose the perturbation $\delta\boldsymbol{\mu}$ with the opposite sign. Then, the topological gradient can be used as an indicator function defining a descent direction to reduce the value of the cost function. As will be shown in the next sections, this information can be used to develop fast algorithms for tomography reconstruction.

Cost Function Definition

Let us now define an appropriate cost function $\Psi(\boldsymbol{\mu})$ for the problem beforehand. As we want to find a $\boldsymbol{\mu}$ that produces the best approximation of the observations obtained from the object ($\mathbf{p}^*$), we propose $\Psi(\boldsymbol{\mu})$ to be the misfit between $\mathbf{p}^*$ and the projection data obtained from the model ($\mathbf{p}$). Namely,

$$\begin{aligned}\Psi(\boldsymbol{\mu}) &= \|\mathbf{p}^* - \mathbf{p}\|^2, \\ &= \|\mathbf{p}^* - (\boldsymbol{\alpha} - \mathbf{K}\boldsymbol{\mu})\|^2. \qquad (5)\end{aligned}$$

On the same bases, the perturbed cost function $\Psi(\boldsymbol{\mu}_T)$ is defined as

$$\Psi(\boldsymbol{\mu}_T) = \|\mathbf{p}^* - (\boldsymbol{\alpha} - \mathbf{K}\boldsymbol{\mu}_T)\|^2. \qquad (6)$$

From these elements we can calculate an explicit formula for the topological gradient.

Topological Gradient Calculation

In this fully discrete case, to obtain an expression for the topological gradient we first calculate the total variation of the cost function associated to a perturbation $\delta\boldsymbol{\mu}$. That is, subtracting Eqs. (5) and (6), we obtain

$$\begin{aligned}\Psi(\boldsymbol{\mu}_T) &= \Psi(\boldsymbol{\mu}) \\ &+ \mathbf{K}^T(2\mathbf{p} - 2\mathbf{p}^* + \mathbf{K}\delta\boldsymbol{\mu}) \cdot \delta\boldsymbol{\mu} \\ &= \Psi(\boldsymbol{\mu}) \\ &+ 2\mathbf{K}^T(\mathbf{p}^* - \mathbf{p}) \cdot \delta\boldsymbol{\mu} + O(\delta\boldsymbol{\mu}). \qquad (7)\end{aligned}$$

Then, rearranging this expression and according to Eq. (4), the topological gradient is given by

$$\begin{aligned}\mathbf{g}\Psi &= 2\mathbf{K}^T(\mathbf{p}^* - \mathbf{p}) \\ &= 2\mathbf{K}^T(\mathbf{p}^* - (\boldsymbol{\alpha} - \mathbf{K}\boldsymbol{\mu})). \qquad (8)\end{aligned}$$

It is easy to notice that $\mathbf{g}\Psi$ is a vector of N components and that the value $g_i\Psi$ indicate the sensitivity of the cost function to a small perturbation in μ_i. Is also important to remark that the topological gradient does not depend on the perturbation, but instead, indicates the direction the perturbation should be made for every cell.

NUMERICAL RESULTS

Using the topological gradient (Eq. (8)) as an indicator function, we can devise an iterative algorithm (Algorithm 1) that allows us to find an approximate solution for the above mentioned problem.

In this case, and for the sake of simplicity in the numerical examples, we assume that a cell might be either intersected or not ($K_{ij} \in \{0,1\}$). That is, if the center of a particular cell is intersected by the ray then K is one for it (Fig. 2(c)), and zero otherwise

$$K_{ij} = \begin{cases} 1, & \text{ray } j \text{ intersects the center of cell } i; \\ 0, & \text{otherwise.} \end{cases}$$

The measurement vector $\mathbf{p}^*$ is the input data to the algorithm, obtained from the object being reconstructed. The matrix $\mathbf{K}$ is easily computed given that the projection directions are known. A short comment should be made on $\delta\boldsymbol{\mu}$, that does not need to be constant and can be adjusted during algorithm evolution to speed up convergence and provide a more accurate result. In particular, for the results presented next $\delta\boldsymbol{\mu}$ was decreased when oscillations were detected.

Algorithm 1 Image reconstruction based on fully discrete version of the topological derivative

Require: Projection data $\mathbf{p}^*$, matrix $\mathbf{K}$, step size $\delta\boldsymbol{\mu}$ and tol.
Ensure: The reconstructed image $\boldsymbol{\mu}$.
  set $\boldsymbol{\mu}^0 = \mathbf{0}$, $t = 0$, Stop = FALSE
  while Stop = FALSE **do**
    compute $\mathbf{g}\Psi$ using Eq. (8)
    for every cell i **do**
      if $g_i\Psi < 0$ **then**
        $\mu_i^{t+1} = \mu_i^t + \delta\mu_i$
      else
        $\mu_i^{t+1} = \mu_i^t - \delta\mu_i$
      end if
    end for
    if $|\Psi(\boldsymbol{\mu}^t) - \Psi(\boldsymbol{\mu}^{t+1})| > tol$ **then**
      $t = t + 1$
    else
      Stop = TRUE, $\boldsymbol{\mu} = \boldsymbol{\mu}^t$
    end if
  end while

On the computational cost of the algorithm, each iteration requires two matrix-vector products ($N*M$) and two vector sums (N) for the topological gradient computation, a run over vector $\boldsymbol{\mu}^t$ to find $\boldsymbol{\mu}^{t+1}$ (M) and the computation of $\Psi(\boldsymbol{\mu}^{t+1})$ that involves a matrix-vector product ($N*M$). Then, the computational cost of the algorithm is governed by $O(N*M)$.

Test 1 - Standard reconstruction

In order to show the performance of this novel reconstruction algorithm, some results are shown. The projection data $\mathbf{p}^*$ was artificially generated from the original data $\boldsymbol{\mu}^*$ shown in Fig. 3(a) (256 levels grayscale image of size 200×204, $N = 200*204$). From this data, different sets of projection data were generated for $D = 8,\ 16,\ 32$ and 64. The projection data obtained is also shown in Fig. 3.

From these projections, an approximation of the original data $\boldsymbol{\mu}$ was reconstructed using the proposed reconstruction method (Algorithm 1). In this cases was considered $M = (200 + 204) * D$ being enough to capture the complete image projection. As can be seen (Figs. 4(a), 4(b), 4(c) and 4(d) for $D = 8,\ 16,\ 32$ and 64 respectively), the reconstructed result presents good quality even for small number of directions.

In Fig. 5 is presented the behavior of the cost function during the iterative process. As can be seen, the algorithm stabilizes very fast. In approximately 50 iterations the cost function has almost stabilized and very close to 0, meaning that the solution is very close.

(a) Original data.

(b) 8 directions.

(c) 16 directions.

(d) 32 directions.

(e) 64 directions.

Figure 3: Projection data for 8, 16, 32 and 64 directions.

Test 2 - Reconstruction with noise

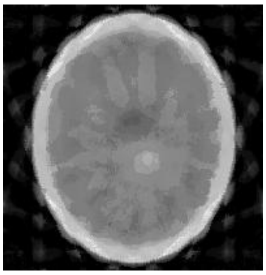
In order to test the robustness of the proposed reconstruction method, in this section are presented some results for different levels of additive noise in the measurement data. The model assumed for the noise polluted measurement $\mathbf{p}^n$ is

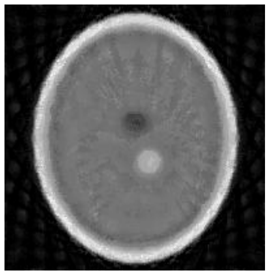
$$\mathbf{p}^n = \mathbf{p}^* + \eta(p) \qquad (9)$$

where η is a zero mean Gaussian distributed random variable, considered as noise in the signal, and

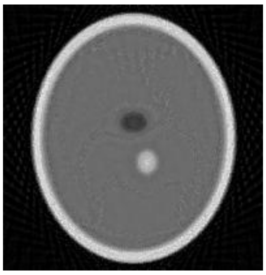
$$p = mean|\mathbf{p}^n - \mathbf{p}^*|.$$

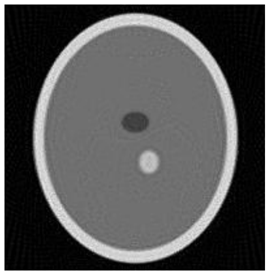
Then, for a signal of strength 100 and $p = 5$, we consider a 5% noise.





(a) Result for $D = 8$. (b) Result for $D = 16$.





(c) Result for $D = 32$. (d) Result for $D = 64$.

Figure 4: Results for the $\mathbf{g}\Psi$ reconstruction method.

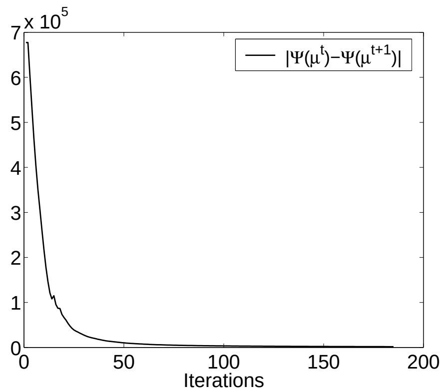
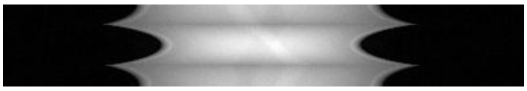
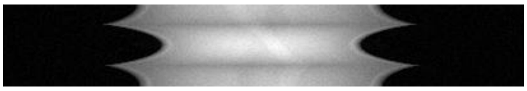


Figure 5: Cost function evolution - Standard reconstruction.

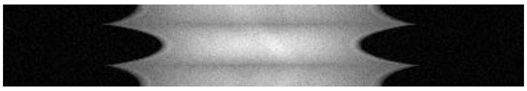
The polluted projection data vector $\mathbf{p}^n$ was created adding white Gaussian noise to $\mathbf{p}^*$ with the model described in (9). The noise added was $p = 1\%, 2\%, 3\%$ and 6%. Figure 6 presents the projection data (always with 64 directions) polluted with noise.



(a) 1% WGN.



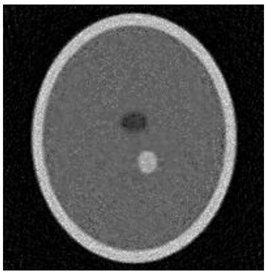
(b) 2% WGN.



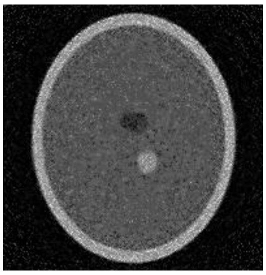
(c) 3% WGN.

(d) 6% WGN.

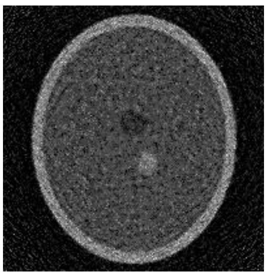
Figure 6: Projection data for 1%, 2%, 3% and 6% WGN.



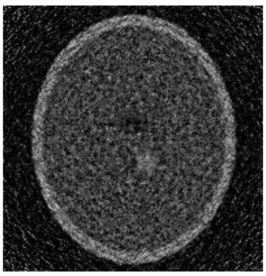
(a) Result 1% WGN.



(b) Result 2% WGN.



(c) Result 3% WGN.



(d) Result 6% WGN.

Figure 7: Results for the $\mathbf{g}\Psi$ reconstruction method from noise data.

As before, the polluted data was used to find an approximation of the original data $\boldsymbol{\mu}$ with Algorithm 1. As can be seen (Figs. 6(a), 6(b), 6(c) and 6(d) for $D = 8,\ 16,\ 32$ and 64 respectively), the $\mathbf{g}\Psi$ method obtains good quality results even for intense noise.

Figure 8 presents the evolution of the cost function. In this case can be observed that the cost function does not converge to zero, but still stabilizes in few iterations (around 70).

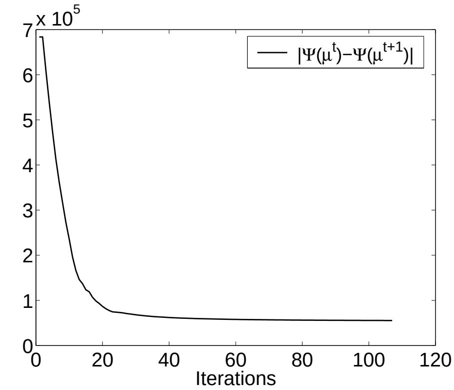


Figure 8: Cost function evolution - Reconstruction with noise.

CONCLUSIONS

In this work the image reconstruction problem was addressed for a simplified tomography model considering the well established concept of topological gradient. The basic idea of the model is that a specific cell is taken into account only if is intersected by a ray, not considering the distance that the ray passes trough it. With this in mind, the problem consists in reconstructing the attenuation coefficient of each cell based on projection information acquired from the real object.

To this end, a cost function that accounts for the misfit between the measure obtained from the real object $\mathbf{p}^*$ and the measure computed using the model $\mathbf{p}$ was used. The model measurement is associated to an approximation $\boldsymbol{\mu}$ of the attenuation coefficient of the real object $\boldsymbol{\mu}^*$. Using the topological expansion, was possible to find an expression for the sensitivity of this cost function to small perturbations in $\boldsymbol{\mu}$. This expression, called topological gradient, was used as an indicator function to find the best places were these perturbations should be introduced. The topological gradient was used to devise an iterative reconstruction algorithm. The proposed algorithm was used to reconstruct artificial data from different number of projections and from noisy data. In all cases good quality results were obtained, even for considerably large levels of noise.

As a final remark we may state that the topological expansion offers a new way to treat the tomography reconstruction problem providing easy, fast and robust reconstruction algorithms.

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