Inverse Problems, Design and Optimization Symposium
Miami, Florida, USA, April 16-18, 2007

TOPOLOGICAL SENSITIVITY ANALYSIS FOR SOURCE PERTURBATION IN TRANSIENT PROBLEMS

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ABSTRACT

In this work we calculate the topological derivative for a quite general heat transfer problem when perturbing the reactive coefficient and the source term as well. This is obtained for two cost functionals, one depending upon a given function on a portion of the boundary and the other based on a Kohn–Vogelius criterion. Then, we use this expression as an indicator function in order to devise an iterative algorithm to apply it in the context of an optimization problem and of an inverse problem.

INTRODUCTION

As it is well known, the topological derivative provides the sensitivity of a given cost functional with respect to an infinitesimal domain perturbation, for instance, an insertion of a hole, an inclusion or a coefficient discontinuous variation at a given point $\hat{\mathbf{x}}$. In case of stationary problems several approaches have been considered for dealing with the calculation of the topological derivative [Céa et al., 2000; Novotny et al., 2003; Sokolowski and Żochowski, 1999]. Nevertheless, there exists very little literature concerning this topic involving transient problems [Amstutz et al., 2006].

On one hand, the topological derivative has been widely used as an descent direction indicator within the context of optimization problems in order to find an optimal solution (in some sense given by a cost functional), by inserting holes or even changing material properties. On the other hand, the use of the topological derivative has been extended to the context of inverse problems by using it as an indicator of which regions should have the topology (or the material property) altered so as to minimize a conveniently stated cost functional. In this way, it would be possible to identify regions whose topology (or material properties) is perturbed by measuring the solution on the boundary. In this way we may mention at least two approaches in order to build a cost functional from measured data: (i) the classical one consists in building a cost functional depending directly upon the information on the boundary and of the direct problem, and (ii) the second one based on a Kohn–Vogelius criterion, that is using the information available on the boundary to set up an auxiliary problem which is used to build a cost functional depending upon the solution of this auxiliary problem and of the direct problem. All this have been done for stationary problems.

In [Amstutz et al., 2006], a rather general class of problems of transient nature as the heat transfer problem and the wave propagation problem are tackled, calculating the topological derivative when all terms in the equation are perturbed. However, in this work it is not considered the existence of convective terms in obtaining the topological derivative, nor numerical examples are presented. In the present work we consider the heat transfer problem incorporating the convective term, and we obtain the topological derivative when perturbing the reactive coefficient as well as the source term. In this way it allows us to quantify the sensitivity of a given cost functional to the infinitesimal alteration of those parameters in a discontinuous manner at a given location $\hat{\mathbf{x}}$.

Concerning applications, we apply the topological derivative as an indicator function to devise iterative algorithms for two situations: (i) an optimization problem for which we want to find the optimal location of sources such that a given target function is achieved; and (ii) an inverse problem where we look, from measured data, for regions whose source term was perturbed.

THE HEAT TRANSFER PROBLEM

In this section we state the problem under study and also the two cost functionals for which we will calculate the topological derivative. The section ends with the statement of the corresponding adjoint problems defined by the direct problems and the cost functionals.

Statement of the problem

As aforementioned, in this work we consider the heat transfer problem involving convection-diffusion-reaction phenomena in $(0,T)\times\Omega$, $\Omega\in\mathbb{R}^n$ $(n=2,3)$, with boundary Γ. We also consider Dirichlet, Neumann and Robin boundary conditions, over Γ_D, Γ_N and Γ_R respectively. Thus, the variational problem consists in finding for each $t\in(0,T)$ $u\in\mathcal{U}$ such that

$$\int_\Omega\left[\frac{\partial u}{\partial t}v+(\mathbf{v}\cdot\nabla u)v+\alpha\nabla u\cdot\nabla v+\beta uv\right]\mathrm{d}\mathbf{x}+\int_{\Gamma_R}h_c uv\,\mathrm{d}\partial\Gamma=\int_\Omega fv\,\mathrm{d}\mathbf{x}+\int_{\Gamma_R}h_c u_\infty v\,\mathrm{d}\Gamma-\int_{\Gamma_N}\bar{q}v\,\mathrm{d}\Gamma\qquad\forall v\in\mathcal{V},\quad(1)$$

with $u(\mathbf{x},0)=0$ in Ω and where, for each $t\in(0,T)$, it is $f\in H^{-1}(\Omega)$, $\bar{q}\in H^{-1/2}(\Gamma_N)$, $\alpha,\beta\in L^\infty(\Omega)$, $h_c\in L^\infty(\Gamma_R)$, $u_\infty\in H^{1/2}(\Gamma_R)$ and $\mathbf{v}\in[L^\infty(\Omega)]^d$ is a vector field such that $\operatorname{div}\mathbf{v}=0$, and also

$$\begin{aligned}\mathcal{U}&=\{v\in H^1(\Omega);\,v_{|\Gamma_D}(\cdot,t)=\bar{u}(\cdot,t)\},\\ \mathcal{V}&=\{v\in H^1(\Omega);\,v_{|\Gamma_D}(\cdot,t)=0\}.\end{aligned}\qquad(2)$$

Cost functionals

In this section we present the cost functionals used in computations. Those functionals depend on a given function over a mensurable portion of the boundary, say Γ^*. Such a known function will constitute, depending upon the case, the target function in the optimization problem or the measurement over the boundary in the inverse problem.

Firstly, let us consider a given function u_d over a portion of the boundary $\Gamma^*=\Gamma_m$ such that $\Gamma_m\cap\Gamma_D=\emptyset$. Then we set the following cost functional

$$\mathcal{J}_{\Omega 1}(u)=\frac{1}{2}\int_0^T\int_{\Gamma_m}(u-u_d)^2\,\mathrm{d}\Gamma\mathrm{d}t,\qquad(3)$$

where u is the solution of the variational problem (1). Observe that u_d may be, in the general case, time dependent.

Secondly, suppose that u_d and q_d are given functions corresponding to temperature and heat flux, and both specified over portions of the boundary Γ_u and Γ_q respectively, such that $\overline{\Gamma_u\cup\Gamma_q}=\Gamma$ and $\Gamma_u\cap\Gamma_q=\emptyset$. Then, let us consider the following cost functional based on the idea of a Kohn–Vogelius criterion depending upon the information available over the boundary

$$\mathcal{J}_{\Omega 2}(u,u^A)=\frac{1}{2}\int_0^T\int_\Omega(u-u^A)^2\,\mathrm{d}\mathbf{x}\mathrm{d}t,\qquad(4)$$

where u is the solution of the variational problem (1), whereas u^A is the solution of the following auxiliary variational problem that consists in finding for each $t\in(0,T)$ $u^A\in\mathcal{W}$ such that

$$\int_\Omega\left[\frac{\partial u^A}{\partial t}v+(\mathbf{v}\cdot\nabla u^A)v+\alpha\nabla u^A\cdot\nabla v+\beta u^A v\right]\mathrm{d}\mathbf{x}=\int_\Omega fv\,\mathrm{d}\mathbf{x}-\int_{\Gamma_q}q_d v\,\mathrm{d}\Gamma\qquad\forall v\in\mathcal{X},\quad(5)$$

with $u^A(\mathbf{x},0)=0$ in Ω and where, for each $t\in(0,T)$, $f,\alpha,\beta,\mathbf{v}$ are as in problem (1), also it is

$$\mathcal{W}=\{v\in H^1(\Omega);\,v_{|\Gamma_u}(\cdot,t)=u_d(\cdot,t)\},\qquad(6)$$

$$\mathcal{X}=\{v\in H^1(\Omega);\,v_{|\Gamma_u}(\cdot,t)=0\}.\qquad(7)$$

In this manner we build information on the whole domain Ω from the information available on the boundary. Then, we measure the gap between the solution of problem (1) and the information over the boundary in the indirect sense given by cost functional (4). It is worth noting that, with this approach, we avoid asking for more regularity for the solution in case of building a cost functional involving an integral term of the type $\int_0^T\int_{\Gamma_q}(-\alpha\nabla u\cdot q_d)^2\,\mathrm{d}\Gamma\mathrm{d}t$ for incorporating information concerning the heat flux over the boundary. Nonetheless, the disadvantage here is that the computational cost has been clearly increased.

Adjoint problems

Once we have introduced the involved problems together with the cost functionals we are in position to set up the corresponding adjoint problems. These problems are presented here for sake of brevity and in view of what will be introduced in the next section.

It is well–known that, for transient problems, the corresponding adjoint problem is a time–reversal problem, that is, a final–boundary value problem. For the heat transfer problem with convection–diffusion–reaction phenomena the adjoint problem corresponding to cost functional (3) consists in finding for each $t\in(0,T)$ $\lambda_1\in\mathcal{Y}$ such that

$$\int_\Omega\left[\frac{\partial\lambda_1}{\partial t}\eta+(\mathbf{v}\cdot\nabla\lambda_1)\eta-\alpha\nabla\lambda_1\cdot\nabla\eta-\beta\lambda_1\eta\right]\mathrm{d}\mathbf{x}-\int_{\Gamma_R}h_c\lambda_1\eta\,\mathrm{d}\Gamma-\int_{\Gamma_N\cup\Gamma_R}(\mathbf{v}\cdot\mathbf{n})\lambda_1\eta\,\mathrm{d}\Gamma=\int_{\Gamma_m}(u_d-u)\eta\,\mathrm{d}\Gamma\qquad\forall\eta\in\mathcal{Y},\quad(8)$$

with $\lambda_1(\mathbf{x},T)=0$ in Ω for convenience, and all the other parameters as defined in the above, also it is

$$\mathcal{Y}=\{\eta\in H^1(\Omega);\eta_{|\Gamma_D}(\cdot,t)=0\}. \qquad (9)$$

Here, λ_1 is the adjoint state of function u.

On the other hand, since cost functional (4) depends upon two functions, u and u^A, that must satisfy, in turn, variational problems (1) and (5), we have two adjoint problems as follows: (i) for each $t\in(0,T)$ find $\lambda_2\in\mathcal{Y}$ such that

$$\begin{aligned}\int_\Omega\left[\frac{\partial\lambda_2}{\partial t}\eta+(\mathbf{v}\cdot\nabla\lambda_2)\eta-\alpha\nabla\lambda_2\cdot\nabla\eta-\beta\lambda_2\eta\right]\mathrm{d}\mathbf{x}\\ -\int_{\Gamma_R}h_c\lambda_2\eta\,\mathrm{d}\Gamma-\int_{\Gamma_N\cup\Gamma_R}(\mathbf{v}\cdot\mathbf{n})\lambda_2\eta\,\mathrm{d}\Gamma=\\ \int_\Omega(u-u^A)\eta\,\mathrm{d}\Gamma\qquad\forall\eta\in\mathcal{Y}, \quad (10)\end{aligned}$$

with $\mathcal{Y}$ defined in (9), and (ii) for each $t\in(0,T)$ find $\lambda^A\in\mathcal{X}$ such that

$$\begin{aligned}\int_\Omega\left[\frac{\partial\lambda^A}{\partial t}\eta+(\mathbf{v}\cdot\nabla\lambda^A)\eta-\alpha\nabla\lambda^A\cdot\nabla\eta\right.\\ \left.-\beta\lambda^A\eta\right]\mathrm{d}\mathbf{x}-\int_{\Gamma_q}(\mathbf{v}\cdot\mathbf{n})\lambda^A\eta\,\mathrm{d}\Gamma=\\ \int_\Omega(u^A-u)\eta\,\mathrm{d}\Gamma\qquad\forall\eta\in\mathcal{X}, \quad (11)\end{aligned}$$

with $\mathcal{X}$ defined in (7). Here, λ_2 and λ^A are the adjoint states of functions u and u^A respectively.

Notice that in the case of functional (3) we obtain as a part of the adjoint problem a Neumann boundary condition from the error on Γ_m (see expression (8)), while in the case of functional (4) we generate a volume source from the error in Ω (see expressions (10)-(11)).

As final remarks it is worth mentioning that the final boundary condition was taken as the zero function in order to simplify the approach. Nonetheless, nothing impedes to consider an arbitrary function. Also, observe that the convective term introduces a Robin boundary condition in the adjoint problem.

TOPOLOGICAL–SHAPE SENSITIVITY ANALYSIS

In this section we present the calculation of the topological derivative for the problem of our concern. This is carried out by means of the theory of shape–sensitivity analysis. We also present what would be called the extension to transient problems of the generalized Eshelby energy momentum tensor.

Preliminaries

Firstly, we need to recall some classical results from continuum mechanics concerning the material derivative of certain quantities. One of the forms of the Reynolds transport theorem expresses that the material derivative of the volume integral of an arbitrary field χ, when there exists a given velocity of change of the boundary $\mathbf{w}_e$, can be written as follows

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\Omega_\tau}\chi_\tau\,\mathrm{d}\mathbf{x}_\tau=\int_{\Omega_\tau}[\dot{\chi}_\tau+\chi_\tau\,\mathrm{div}_\tau\,\mathbf{w}_e]\,\mathrm{d}\mathbf{x}_\tau. \qquad (12)$$

where $\tau\in\mathbb{R}^+$ is the control parameter that governs the magnitude of the velocity of change. Also we need the expressions of the material derivatives of the gradient of a scalar field φ and a vector field $\mathbf{u}$ as follows

$$(\dot{\nabla_\tau\varphi})=\nabla_\tau\dot{\varphi}-\mathbf{L}_\tau^T\nabla_\tau\varphi, \qquad (13)$$
$$(\dot{\nabla_\tau\mathbf{u}})=\nabla_\tau\dot{\mathbf{u}}-(\nabla_\tau\mathbf{u})\mathbf{L}_\tau, \qquad (14)$$

where $\mathbf{L}_\tau=\nabla_\tau\mathbf{w}_e$. Also it will be consider the following identity $\mathbf{I}\cdot\mathbf{L}_\tau=\mathbf{I}\cdot\nabla_\tau\mathbf{w}_e=\mathrm{div}_\tau\,\mathbf{w}_e$.

Sensitivity analysis and the generalized Eshelby energy momentum tensor

We aim at calculating the sensitivity of some cost functional when an infinitesimal perturbation in some parameters is introduced. This can be done in several ways. The approach considered in this work, as mentioned, is based on the theory of shape–sensitivity analysis. We will present the guidelines in obtaining the topological derivative for the case of cost functional (3). For functional (4) the result follows in a completely analogous manner. Let us consider first an existent portion of the domain, say $\omega_\varepsilon\subset\Omega_\varepsilon$, with Ω_ε denoting the domain with the region $\omega_\varepsilon=\hat{\mathbf{x}}+\varepsilon\omega$ being ω fixed and containing the origin. This particular region will be characterized by the abrupt change of the source term f and the reaction coefficient β. Thus, we have that these parameters have the following form

$$f(\mathbf{x})=\begin{cases}f_0 & \text{in } \Omega_\varepsilon\setminus\overline{\omega}_\varepsilon\\ f_1 & \text{in } \omega_\varepsilon\end{cases}, \qquad (15)$$

$$\beta(\mathbf{x})=\begin{cases}\beta_0 & \text{in } \Omega_\varepsilon\setminus\overline{\omega}_\varepsilon\\ \beta_1 & \text{in } \omega_\varepsilon\end{cases}, \qquad (16)$$

where f_0, f_1, β_0 and β_1 are arbitrary functions whose values differ on the boundary $\partial\omega_\varepsilon$. In this work we consider only constant functions for the sake of brevity. Therefore, we have a jump in these quantities on such a boundary $\partial\omega_\varepsilon$.

Hence, we divide the domain Ω_ε according to the portion ω_ε, and calculate the sensitivity of the cost functional when perturbing the shape of each portion of the domain as shown in figure 1.

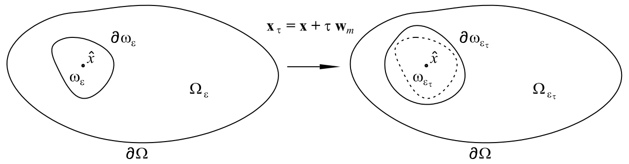

Figure 1. Perturbation of domain Ω_ε.

In order to obtain the sensitivity of cost functional $\mathcal{J}_{\Omega_\varepsilon 1}$ we build the corresponding Lagrangian functional

$$\mathcal{L}_{\Omega_\varepsilon 1}(\hat{u},\hat{\lambda}_1) = \mathcal{J}_{\Omega_\varepsilon 1}(\hat{u}) + \int_0^T \mathcal{R}_{\Omega_\varepsilon}(\hat{u},\hat{\lambda}_1)\,\mathrm{d}t, \qquad (17)$$

where $\mathcal{R}_{\Omega_\varepsilon}$ denotes the variational problem (1) when existing region ω_ε. When $\hat{u} = u_\varepsilon$ is in fact the solution of such a variational problem, with f and β defined as in (15) and (16) respectively, we have

$$\mathcal{R}_{\Omega_\varepsilon}(u_\varepsilon, v) = 0 \qquad \forall v \in \mathcal{V}. \qquad (18)$$

With these considerations we are able to perform a perturbation of the domain and to calculate the sensitivity of the Lagrangian functional. In this manner we circumvent the fact that the involved functions are restricted to satisfy the corresponding variational problems. Indeed, it is well–known that

$$\dot{\mathcal{L}}_{\Omega_\varepsilon 1}(\hat{u},\hat{\lambda}_1)\Big|_{\substack{\hat{u}=u_\varepsilon \\ \hat{\lambda}_1=\lambda_{\varepsilon 1}}} = \dot{\mathcal{J}}_{\Omega_\varepsilon 1}(u_\varepsilon). \qquad (19)$$

where $\dot{()}$ denotes the sensitivity, and $\lambda_{\varepsilon 1}$ is the solution to the adjoint problem (8) when the parameters f and β are defined as in (15)–(16).

With these expressions and the relations given by the continuum mechanics already presented we can write the sensitivity of the cost functional as follows

$$\dot{\mathcal{J}}_{\Omega_\varepsilon 1}(u_\varepsilon) = \int_0^T \int_{\Omega_\varepsilon} \Sigma_{\varepsilon 1} \cdot \nabla \mathbf{w}_m \,\mathrm{d}\mathbf{x}\mathrm{d}t + \int_0^T \int_{\Omega_\varepsilon} \mathbf{b}_{\varepsilon 1} \cdot \mathbf{w}_m \,\mathrm{d}\mathbf{x}\mathrm{d}t, \qquad (20)$$

where $\mathbf{w}_m$ is the material representation of the shape change velocity $\mathbf{w}_e$. Also, it appears the tensor $\Sigma_{\varepsilon 1}$ that is identified as the generalized Eshelby energy momentum tensor [Gurtin, 2000]. The form of this tensor in this case is the following

$$\Sigma_{\varepsilon 1} = \theta_{\varepsilon 1}\mathbf{I} + \mathbf{K}_{\varepsilon 1}, \qquad (21)$$

with $\mathbf{I}$ the identity tensor and

$$\theta_{\varepsilon 1} = \frac{\partial u_\varepsilon}{\partial t}\lambda_{\varepsilon 1} + (\mathbf{v}\cdot\nabla u_\varepsilon)\lambda_{\varepsilon 1} + \alpha \nabla u_\varepsilon \cdot \nabla \lambda_{\varepsilon 1} + \beta u_\varepsilon \lambda_{\varepsilon 1} - f\lambda_1, \qquad (22)$$

$$\mathbf{K}_{\varepsilon 1} = -\alpha\big(\nabla u_\varepsilon \otimes \nabla\lambda_{\varepsilon 1} + \nabla\lambda_{\varepsilon 1} \otimes \nabla u_\varepsilon\big) - (\lambda_{\varepsilon 1}\nabla u_\varepsilon) \otimes \mathbf{v}, \qquad (23)$$

while vector $\mathbf{b}_{\varepsilon 1}$ is given by

$$\mathbf{b}_{\varepsilon 1} = (\nabla \mathbf{v})^T \nabla u_\varepsilon \lambda_{\varepsilon 1}, \qquad (24)$$

since in our case we have considered an arbitrary velocity $\mathbf{v}$. In case of constant velocity $\mathbf{v}$ or even in absence of convective effects we have $\mathbf{b}_{\varepsilon 1} = 0$.

It is interesting to explore some properties of the tensor $\Sigma_{\varepsilon 1}$. In fact, this tensor plays an important role within what is known as the balance of configurational forces when a perturbation of the kind performed here is introduced. Under suitable regularity assumptions and using basic tensorial identities it is quite straightforward to prove that in this case of a transient problem the balance of configurational forces is given by the following equation

$$\int_0^T \operatorname{div}\Sigma_{\varepsilon 1}\,\mathrm{d}t = \int_0^T \mathbf{b}_{\varepsilon 1}\,\mathrm{d}t \qquad \text{in } \Omega_\varepsilon. \qquad (25)$$

With this property at hand and considering that

$$\mathbf{w}_m = \begin{cases} 0 & \text{on } \Gamma, \\ \mathbf{n} & \text{on } \partial\omega_\varepsilon, \end{cases} \qquad (26)$$

where $\mathbf{n}$ is the unit outward normal to ω_ε, we can rewrite the sensitivity (20) by simply using the Green formula as follows

$$\dot{\mathcal{J}}_{\Omega_\varepsilon 1}(u_\varepsilon) = \int_0^T \int_{\partial\omega_\varepsilon} [\![\Sigma_{\varepsilon 1}]\!]\mathbf{n}\cdot\mathbf{n}\,\mathrm{d}\Gamma\mathrm{d}t, \qquad (27)$$

where $[\![\cdot]\!]$ denotes the jump of the $\Sigma_{\varepsilon 1}$ tensor as a result of the jumps introduced in f and β. It follows that the sensitivity of the cost functional can be written as a function of the flux of the jump of the Eshelby tensor across the boundary which is being perturbed. With the expression of $\Sigma_{\varepsilon 1}$ it is easy to show that

$$[\![\Sigma_{\varepsilon 1}]\!]\mathbf{n}\cdot\mathbf{n} = (\beta_1 - \beta_0)u_\varepsilon\lambda_{\varepsilon 1} - (f_1 - f_0)\lambda_{\varepsilon 1}, \qquad (28)$$

where it has been considered that α and also that the flux $-\alpha\frac{\partial u_\varepsilon}{\partial \mathbf{n}}$ are continuous on $\partial\omega_\varepsilon$. Substituting this expression into (27) we arrive at the following

$$\dot{\mathcal{J}_{\Omega_\varepsilon 1}}(u_\varepsilon) = \int_0^T \int_{\partial\omega_\varepsilon} \big[(\beta_1-\beta_0)u_\varepsilon\lambda_{\varepsilon 1} - (f_1-f_0)\lambda_{\varepsilon 1}\big]\,\mathrm{d}\Gamma\mathrm{d}t, \quad (29)$$

that constitutes the expression for the sensitivity of the cost functional (3) when a jump over $\partial\omega_\varepsilon$ occurs in parameters f and β.

In a completely analogous way we found for the cost functional $\mathcal{J}_{\Omega_\varepsilon 2}$ that the Eshelby tensor has the same form as in (21) with

$$\begin{aligned}\theta_{\varepsilon 2} = {} & (u_\varepsilon-u_\varepsilon^A)^2 + \frac{\partial u_\varepsilon}{\partial t}\lambda_{\varepsilon 2} + (\mathbf{v}\cdot\nabla u_\varepsilon)\lambda_{\varepsilon 2} \\ & + \alpha\nabla u_\varepsilon\cdot\nabla\lambda_{\varepsilon 2} + \beta u_\varepsilon\lambda_{\varepsilon 2} - f\lambda_{\varepsilon 2} \\ & + \frac{\partial u_\varepsilon^A}{\partial t}\lambda_\varepsilon^A + (\mathbf{v}\cdot\nabla u_\varepsilon^A)\lambda_\varepsilon^A \\ & + \alpha\nabla u_\varepsilon^A\cdot\nabla\lambda_\varepsilon^A + \beta u_\varepsilon^A\lambda_\varepsilon^A - f\lambda_\varepsilon^A, \end{aligned} \quad (30)$$

$$\begin{aligned}\mathbf{K}_{\varepsilon 2} = {} & -\alpha\big(\nabla u_\varepsilon\otimes\nabla\lambda_{\varepsilon 2} + \nabla\lambda_{\varepsilon 2}\otimes\nabla u_\varepsilon\big) \\ & - \alpha(\nabla u_\varepsilon^A\otimes\nabla\lambda_\varepsilon^A + \nabla\lambda_\varepsilon^A\otimes\nabla u_\varepsilon^A) \\ & - (\lambda_{\varepsilon 2}\nabla u_\varepsilon + \lambda_\varepsilon^A\nabla u_\varepsilon^A)\otimes\mathbf{v}, \end{aligned} \quad (31)$$

while vector $\mathbf{b}_{\varepsilon 2}$ is

$$\mathbf{b}_{\varepsilon 2} = (\nabla\mathbf{v})^T[\nabla u_\varepsilon\lambda_{\varepsilon 2} + \nabla u_\varepsilon^A\lambda_\varepsilon^A]. \quad (32)$$

The sensitivity in this case takes the following form

$$\dot{\mathcal{J}_{\Omega_\varepsilon 2}}(u_\varepsilon,u_\varepsilon^A) = \int_0^T \int_{\partial\omega_\varepsilon} \big[(\beta_1-\beta_0)(u_\varepsilon\lambda_{\varepsilon 2} + u_\varepsilon^A\lambda_\varepsilon^A) - (f_1-f_0)(\lambda_{\varepsilon 2}+\lambda_\varepsilon^A)\big]\,\mathrm{d}\Gamma\mathrm{d}t. \quad (33)$$

Here $\lambda_{\varepsilon 2}$ and λ_ε^A are the solutions to the adjoint problems (10)-(11) when the parameters f and β are defined as in (15)-(16).

The topological derivative

The next and final step in order to obtain the expression of the topological derivative is to calculate the limit when $\varepsilon \to 0$. Indeed, the topological derivative, in this problem, gives the sensitivity of the cost functionals when parameters within an infinitesimal region acquire a value that differs from the rest of the domain in a discontinuous fashion, or, in other words, when the region ω_ε with its own characteristics is actually created.

In [Novotny et al., 2003] it is proven that the topological derivative at point $\hat{\mathbf{x}}$, denoted by $\mathcal{D}_\mathcal{T}\mathcal{J}_{\Omega 1}(\hat{\mathbf{x}})$ for the cost functional (3), can be obtained by calculating the following limit

$$\mathcal{D}_\mathcal{T}\mathcal{J}_{\Omega 1}(\hat{\mathbf{x}}) = \lim_{\varepsilon\to 0}\frac{\dot{\mathcal{J}_{\Omega_\varepsilon 1}}(u_\varepsilon)}{h'(\varepsilon)}, \quad (34)$$

where h is a monotonically descent function that approaches from zero when $\varepsilon \to 0$.

It is not difficult to achieve expansions of the form

$$\begin{aligned} u_\varepsilon &= u + O_u(\varepsilon^{n/2}), \\ \lambda_{\varepsilon 1} &= \lambda_1 + O_\lambda(\varepsilon^{n/2}), \end{aligned} \quad (35)$$

where u and λ_1 are the solutions to problems (1)-(8) when no perturbation is introduced in parameters f and β and n is the spatial dimension. In fact, these expansions can be given in the sense of some norm of a suitable Sobolev space, then, using compactness arguments under some regularity assumptions we reach the desired result. Introducing these expansions into expression (29) and into (34), using the localization's theorem and considering $h'(\varepsilon) = |\partial\omega_\varepsilon|$ the measure of the boundary of ω_ε, we obtain

$$\mathcal{D}_\mathcal{T}\mathcal{J}_{\Omega 1}(\hat{\mathbf{x}}) = \int_0^T \big[(\beta_1-\beta_0)u(\hat{\mathbf{x}},t)\lambda_1(\hat{\mathbf{x}},t) - (f_1-f_0)\lambda_1(\hat{\mathbf{x}},t)\big]\,\mathrm{d}t, \quad (36)$$

that is the expression of the topological derivative for cost functional (3).

Analogously, we obtain for the cost functional (4) the following expression for the topological derivative

$$\begin{aligned}\mathcal{D}_\mathcal{T}\mathcal{J}_{\Omega 2}(\hat{\mathbf{x}}) = \int_0^T & \big[(\beta_1-\beta_0)u(\hat{\mathbf{x}},t)\lambda_2(\hat{\mathbf{x}},t) \\ & + (\beta_1-\beta_0)u^A(\hat{\mathbf{x}},t)\lambda^A(\hat{\mathbf{x}},t) \\ & - (f_1-f_0)(\lambda_2(\hat{\mathbf{x}},t)+\lambda^A(\hat{\mathbf{x}},t))\big]\,\mathrm{d}t, \end{aligned} \quad (37)$$

where u, u^A, λ_2 and λ^A are the solutions of variational problems (1), (5), (10) and (11) respectively, when $f = f_0$ and $\beta = \beta_0$ in Ω.

NUMERICAL ASSESSMENTS

In this section we present two examples for which we apply the concept of the topological derivative. In our examples we limit ourselves to perturbing the source term f. So, in expressions (36)–(37) we have $\beta_1 = \beta_0$.

Example 1:

Consider the situation shown in figure 2. We have a given portion of the boundary $\Gamma^* = \Gamma_R$ for which we have a given time dependant function u_d. The meaning of this function depends on whether it is an optimization or an inverse problem. Such a function is shown in figure 3. Also it is $f = 1$, $\alpha = 1$, $\beta = 0$, $\mathbf{v} = 0$, $h_c = 1$, $u_\infty = 0.1$, $T = 0.1$ and $\bar{u} = 0$ on Γ_D. So it is $\Gamma_N = \emptyset$. Three constraints are considered for this problem:

-The perturbations of the source term must be placed on the dotted line (section A–A).

-The perturbations are carried out with a factor of 20. That is, starting with $f = 1$ we may pass to $f = 0.05$ or to $f = 20$, being those the only possibilities considered.

-The size of the perturbation is at least 0.02.

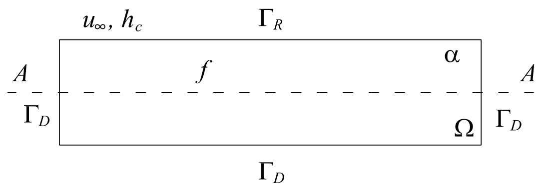

Figure 2. Setting of example 1.

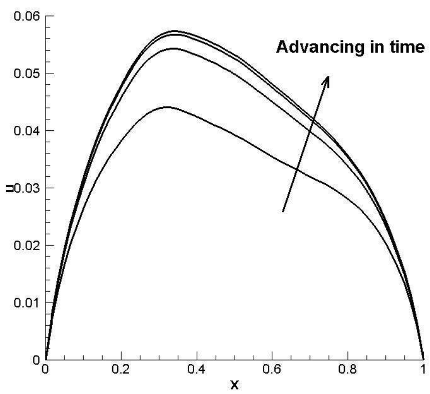

Figure 3. Curve on Γ^*.

Then, using the topological derivative as an indicator function we may devise the following iterative process (here presented for cost functional $\mathcal{J}_1$):

(a) Solve variational problems (1)–(8) computing u and λ_1.
(b) Compute the topological derivative with expression (36), but without considering the factor $f_1 - f_0$.
(c) Look for those locations, on section A–A, where $\mathcal{D}_T \mathcal{J}_{\Omega 1}(\hat{\mathbf{x}})$ takes its minimum and maximum values.
(d) Set $f_1 = 20 f_0$ ($f_1 = 0.05 f_0$) in the location of minimum (maximum) value of $\mathcal{D}_T \mathcal{J}_{\Omega 1}(\hat{\mathbf{x}})$.
(e) Consider now the altered function f of step (d) into variational problems (1)–(8) and go back to step (a).

For cost functional $\mathcal{J}_2$ the iterative process is completely similar, but using variational problems (1)–(5) and (10)–(11) in step (a), and the expression (37) in step (c).

Both cases, the use of the topological derivative in an optimization and in an inverse problems, are depicted in the following.

i. Optimization problem:
In this case function u_d is a given target function we want to achieve. Then, we want to found a configuration of sources such that cost functional (3) (or (4)) is minimized. In this case the set of constraints introduced in the above corresponds to design criteria. The minimization process leads, after 22 iterations, to the result shown in figure 4. There we compare the target function u_d (continuous line) with the achieved result u (dashed line) for several time instants.

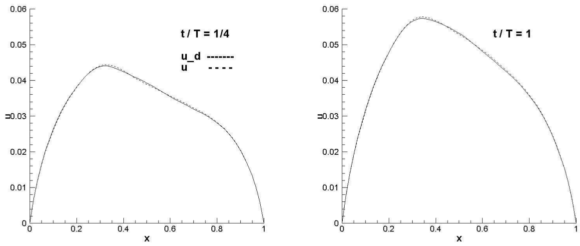

Figure 4. Comparison between the achieved solution (dashed line) and the target function (continuous line) for cost functional $\mathcal{J}_{\Omega 1}$.

In the same way, when considering the cost functional (4) we reach, after 22 iterations, the results shown in figure 5.

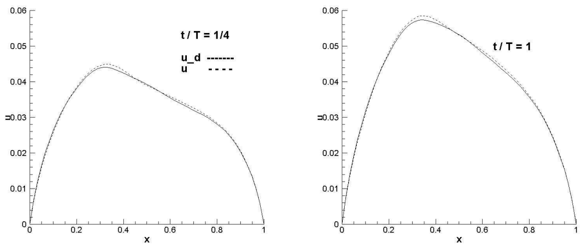

Figure 5. Comparison between the achieved solution (dashed line) and the target function (continuous line) for cost functional $\mathcal{J}_{\Omega 2}$.

ii. Inverse problem
In this case function u_d is a measurement acquired from the real problem including the actual source configuration. The objective here is to try to find the source configuration that gives such a measurement on that portion of the boundary. That is, we want to found a configuration of sources such that cost functional (3)

(or (4)) is minimized. In this case the set of constraints introduced in the above corresponds to additional information we know about the inverse problem. The minimization process leads, after 22 iterations, to the results shown in figure 6. There we show the progress of the source configuration.

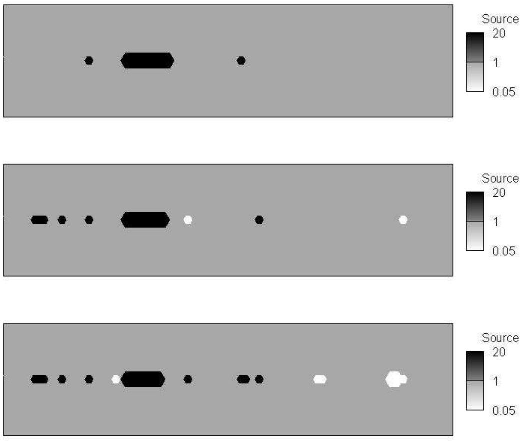

Figure 6. From top to bottom, source configuration in iterations 10, 15 and 22 for cost functional $\mathcal{J}_{\Omega 1}$.

Analogously, with cost functional (4) we reach, after 22 iterations, the results shown in figure 5.

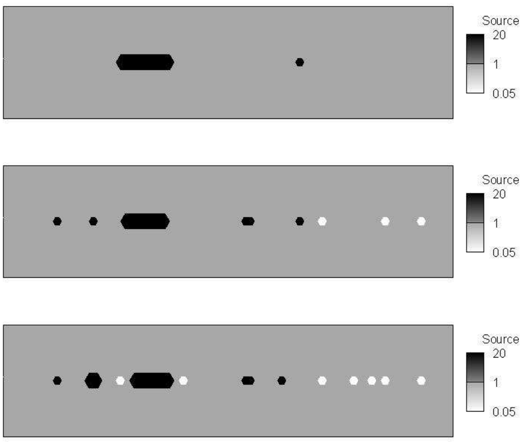

Figure 7. From top to bottom, source configuration in iterations 10, 15 and 22 for cost functional $\mathcal{J}_{\Omega 2}$.

It is interesting to note that in both cases, optimization and inverse problems, the problem we are solving is exactly the same. The difference is on the meaning of the constraints and on the interpretation of the results. Indeed, while for the optimization problem we are interested in the function we achieve on the boundary Γ_m, in the inverse problem we are interested in the final source configuration in order to compare it with the target that is given in figure 8.

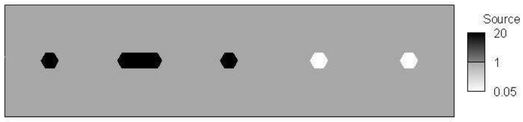

Figure 8. Target source configuration.

In figures 9 and 10 we present the values of the cost functionals for each case. Also, the comparison of the value of cost functional $\mathcal{J}_{\Omega 1}$ between the solution obtained with the proper cost functional, that is u_1, and that obtained with cost functional $\mathcal{J}_{\Omega 2}$, that is u_2 is included in figure 9.

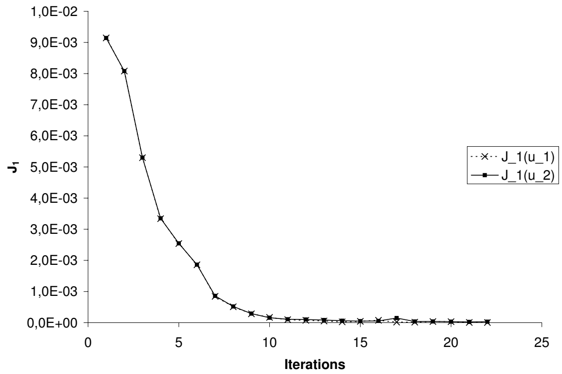

Figure 9. Minimization process through the value of cost functional $\mathcal{J}_{\Omega 1}$.

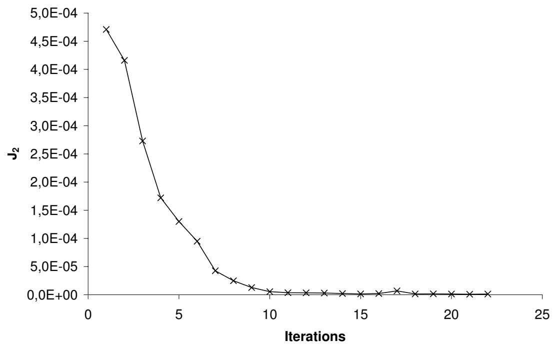

Figure 10. Minimization process through the value of cost functional $\mathcal{J}_{\Omega 2}$.

Example 2:

This example corresponds to the use of the topological derivative in an inverse problem. The goal is to find the location of a region where the source has been altered. In this case we have no restrictions concerning with the location of the source perturbation. We limit ourselves to show the topological derivative computed at the first iteration.

For this example we work with the cost functional (4), and the parameters that define the problem are $\alpha = 1$, $\beta = 1$, $f = 1$, $\mathbf{v} = \frac{\sqrt{2}}{2}\mathbf{e}_x + \frac{\sqrt{2}}{2}\mathbf{e}_y$, $T = 1$, $\bar{q} = 0$ on Γ_N, $\bar{u} = 1$ on Γ_{D1} and $\bar{u} = 0$ on Γ_{D2}. So, it is $\Gamma_R = \emptyset$. Figure 11 shows the setting of the problem, being $L = 1$. For the auxiliary problem we have that u_d is the measurement of the temperature on the portion of the boundary Γ_N and q_d is the measurement of the heat flux on Γ_D, both measurements taken from the real problem including the region with a perturbed value of f, that is a disc of radius $\frac{L}{10}$ centered at point $(\frac{L}{4}, \frac{L}{2})$. In this manner it is $\Gamma_q = \Gamma_D$ and $\Gamma_u = \Gamma_N$.

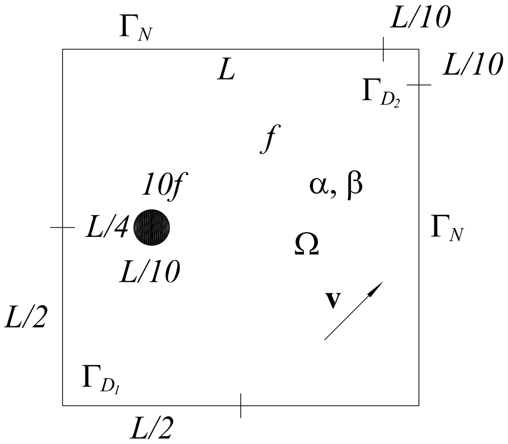

Figure 11. Setting of example 2.

For this example we present in figure 12 the result just for the first iteration of what would be the iterative process presented in the above.

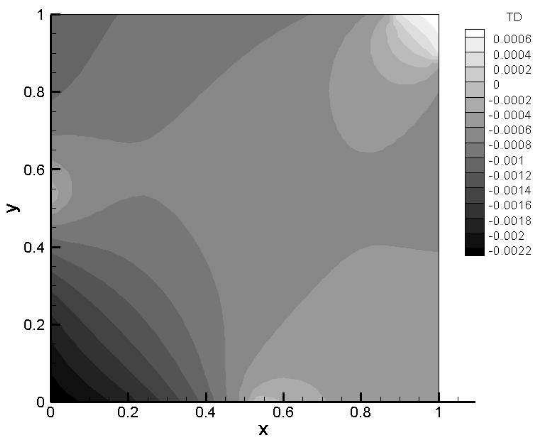

Figure 12. Topological derivative.

In this case, as expected, the minimum value of the topological derivative is achieved on the boundary. In the previous example this drawback, inherent to the problem, was circumvented by considering the value of the topological derivative over section *A–A*.

It is important to observe that the topological derivative is actually the sum of the solutions of both adjoint problems. Therefore, the result is a balance between the adjoint states. It is not difficult to achieve in practice a situation for which one of the adjoint states dominates the value of the topological derivative, as a result of out–of–scale problems. This scaling trouble can be avoided by introducing a suitable real parameter multiplying the term associated with the restriction $\mathcal{R}_\Omega$ in expression (17). This subject is still being studied, and would help in handling problems like the presented in this last example.

CONCLUSIONS

In this work the topological derivative for source and reactive coefficient perturbation was calculated for a further general version of the time–dependent heat transfer problem. Besides, a configurational force balance was obtained for the sensitivity analysis problem and also the generalized Eshelby tensor was identified.

In applications, when using the topological derivative as an indicator function, it showed promising results proving to be a powerful tool in optimization problems. On the other hand, its use as an approach to inverse problems is still a matter of further study.

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