# A variational approach for coupling kinematically incompatible structural models

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**Abstract**

In this work an extended variational framework aimed at properly addressing the coupling of kinematically incompatible structural models is presented. The main goal is to variationally state the theoretical basis to deal with the coupling of structural models with different dimensionality. In this approach, the coupling conditions are naturally derived from the governing variational principle formulated at the continuous level. In particular, the coupling of 3D solid models and 2D shell models, under Naghdi hypothesis, is treated by introducing the corresponding kinematical assumptions into the proposed extended variational principle. Also, the coupling between 3D solid and 1D beam models, under Bernoulli hypothesis, is presented. Finally, a discussion comprising the main conclusions of the work is given.

*Key words:* Incompatible kinematics, Variational formulation, Coupling conditions, Structural models.

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## 1 Introduction

In structural analysis it is common to make use of reduced models in order to represent the main phenomena involved in the problem. Those models are built by taking advantage of the particular form of the geometry of the structural component and of the loading acting on it. In this way, full 3D models can

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be condensed into shells, plates or even beams. This kind of reduction in the dimension of the problem, within the context of primal variational formulations, is given by deeming suitable kinematical assumptions, that is, assuming a particular form for the displacement vector field.

The situation we are interested in involves a structural component whose geometry and loadings take a very general configuration on a given portion of the domain of analysis, for which it is necessary to work with the full 3D model, whereas they take a particular configuration over the rest of the domain, where it is possible to introduce some kinematical restrictions. Research oriented to this field was formerly conducted in the 80's in works that dealt with the problem of junctions between plates, and between plates and 3D elastic bodies [1,4,7]. Lately, the problem was also extended to junctions between shells, while the numerical study of junctions between elastic bodies and plates continued [2,3,9]. In the 90's, some works covered the asymptotic analysis for the coupling between a 3D elastic body and a dimensionally reduced structure [10,13,14]. In all the aforementioned works the problem of performing a junction was analyzed from a very different standpoint than the one presented in this work. Indeed, in the present work the problem of coupling structural components is generalized. It will be seen that it is possible to handle any kind of junction in spite of the possibility of non–linear constitutive behaviors, as well as non–linearities arisen from large displacements. Thus, all those problems treated in the aforesaid literature can be embraced in the ideas developed here. It is worth noting that, up to the authors' knowledge, there is no previous works that deal into a unified continuous variational framework with the coupling between different structural solid models. Previous ideas regarding the coupling of models of different dimensionality from a purely kinematical point of view were explored in [8,17], but they have recently been variationally stated for the fluid flow problem in [5].

The concepts introduced here can also be useful in the context of domain decomposition techniques as a clever alternative approach to formulate a problem with non–matching meshes. Although in this situation there exists kinematical compatibility at the continuous level, this compatibility is lost when passing to the discrete level by introducing different approximations for the different partitions of the domain of analysis. Thus, the theory is also applicable for handling partitioned systems, and from this perspective it involves several well–known methods as those proposed in [15,16].

According to what was said in a previous paragraph, we have a structural component in which, for simplicity but without loss of generality, two possible incompatible kinematics coexist (this number can be arbitrary). Since the involved fields may suffer jumps over the locations where the kinematics changes, the original governing variational principle is not correctly stated. Then, the need for an extended variational formulation can be interpreted as a consequence of working with non–matching underlaying kinematics defined over complementary portions of the domain of analysis. This formulation is mathematically founded on the Lagrange multiplier theory, and can be understood from a mechanical viewpoint by means of introducing, into the original variational principle, some terms related to the jump in the fields, allowing discontinuities to occur over an artificial internal boundary where the kinematics changes. In this manner, the extended governing variational principle yields, as the natural Euler–Lagrange equations, besides the equilibrium equations, the coupling conditions between both models. These natural coupling conditions depend exclusively upon the kinematics adopted for each domain. It will be seen that, according to the way in which the jumps over the internal boundary are introduced, different possibilities regarding the final Euler–Lagrange equations emerge. Moreover, this extended variational principle holds the property of consistency in the following sense: when no difference between the kinematics is considered, the Euler–Lagrange equations ensure exactly the same solution of the original problem without any discontinuities.

Having taken into account the possible occurrence of discontinuities we are in position to perform any kind of kinematical restriction over just a portion of the domain of analysis, reducing the full 3D model to, for example, a beam or a shell model. It is worthwhile to mention here that when taking such a restriction over the displacement field we are also altering the way in which the involved duality product is defined. This aspect entails interesting consequences for the setting of the whole problem, such as changes in the regularity conditions in the part of the domain complementary to that in which the kinematics changed. Then, once the foundations of the coupled problem are well established, as done in the present work, a numerical approximation constitutes the next step for which, given the continuous formulation, any approximation method could eventually be used.

The organization of this paper is as follows. In Section 2 the original and extended variational principles are presented. The coupling between 3D full models and 2D shell models, under Naghdi hypothesis [11,12], is derived in Section 3, while in Section 4 the coupling between 3D full models and 1D beam models, under Bernoulli hypothesis, is obtained. In all cases we formulate the equilibrium problem for a general constitutive behavior, although a simple example involving linear elastic materials in small displacements is presented to prove existence and uniqueness of the solution. The theory is introduced for any material, but the constitutive modelling problem is left aside in this work since it can be managed according to the classical constitutive theory well established for each model. In Section 5 final remarks and general conclusions are given.

## 2 The variational framework

In this section the usual variational principle is recalled, then the extended governing variational formulation is devised in order to allow a discontinuity in the displacement field to occur at a given location of the domain of analysis. As a consequence of that, some relation between the original problem and the extended one has to be assured. Therefore, the added terms play the important role of making compatible both problems in terms of the corresponding Euler–Lagrange equations.

### *2.1 Original variational principle*

Let $\Omega \subset \mathbb{R}^3$ be the domain of analysis, with boundary decomposed as $\Gamma = \Gamma_D \cup \Gamma_N$, such that $\Gamma_D \cap \Gamma_N = \emptyset$. Then, without loss of generality, let us consider the following structural analysis problem neglecting dynamics:

**Problem 1** *Find* $\mathbf{u} \in \boldsymbol{\mathcal{U}}$ *such that*

$$\int_\Omega \boldsymbol{\sigma} \cdot \nabla \mathbf{v} \, \mathrm{d}\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, \mathrm{d}\mathbf{x} + \int_{\Gamma_N} \bar{\mathbf{t}} \cdot \mathbf{v} \, \mathrm{d}\Gamma \qquad \forall \mathbf{v} \in \boldsymbol{\mathcal{V}}, \tag{1}$$

*where*

$$\boldsymbol{\mathcal{U}} = \{\mathbf{u} \in \mathbf{H}^1(\Omega);\, \mathbf{u}_{|\Gamma_D} = \bar{\mathbf{u}}\}, \tag{2}$$

*with* $\boldsymbol{\mathcal{V}}$ *the space obtained from differences between elements of* $\boldsymbol{\mathcal{U}}$*. Also,* $\Omega$ *is the actual deformed configuration of the body, and therefore* $\boldsymbol{\sigma}$ *is the Cauchy stress tensor,* $\mathbf{f}$ *is a volume force,* $\bar{\mathbf{t}}$ *is a traction acting over the Neumann boundary* $\Gamma_N$ *and* $\bar{\mathbf{u}}$ *is a displacement prescribed over the Dirichlet boundary* $\Gamma_D$*.*

Notice that the problem is actually closed once the material behavior is given by specifying the dependence of $\boldsymbol{\sigma}$ on $\mathbf{u}$. This problem is rather general, and remains valid, for example, for large displacements and deformations and non-linear constitutive behavior.

Consider now a smooth artificial internal boundary $\Gamma_a$ that splits the domain $\Omega$ as $\Omega = (\overline{\Omega_1 \cup \Omega_2})^\circ$, where $\Gamma_i$ is the boundary of $\Omega_i$, $i = 1, 2$, and hence $\Gamma_a = \Gamma_1 \cap \Gamma_2$ and $\Gamma = (\Gamma_1 \cup \Gamma_2) \setminus \Gamma_a$, being $\Gamma$ the boundary of $\Omega$. In this manner, it is well–known that the solution of Problem 1, considered now as a pair $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ according to the partition of $\Omega$, satisfies the following conditions over $\Gamma_a$

$$\mathbf{u}_1 = \mathbf{u}_2 \qquad \text{in } \mathbf{H}^{1/2}(\Gamma_a), \tag{3}$$

$$\boldsymbol{\sigma}_1 \mathbf{n}_1 = \boldsymbol{\sigma}_2 \mathbf{n}_1 \qquad \text{in } \mathbf{H}^{-1/2}(\Gamma_a), \tag{4}$$

where $\boldsymbol{\sigma}_i$ is the stress tensor corresponding to the partition $\Omega_i$, $i = 1, 2$, over $\Gamma_a$ whose unit outward normal, seen from $\Omega_1$, is $\mathbf{n}_1$. It is clear that condition (3) follows from the kinematics of the problem as a result of the definition of set $\boldsymbol{\mathcal{U}}$, while condition (4) is a natural consequence of the variational formulation (1).

### *2.2 Extended variational principle*

Consider now the scheme shown in Figure 1, where the decomposition of $\Omega$ performed by the artificial internal boundary $\Gamma_a$ is shown explicitly.

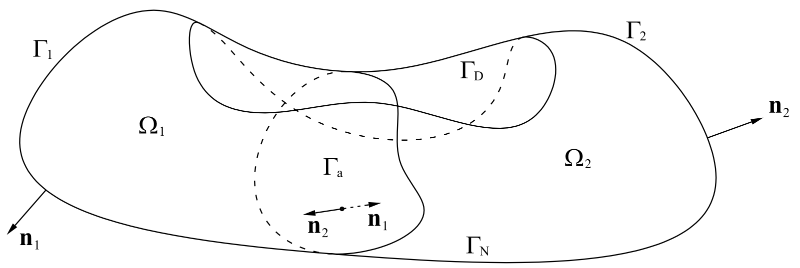


Fig. 1. Decomposition of domain Ω.

Here, we are preparing the theory for introducing, as will be seen in Section 3 and Section 4, suitable kinematical restrictions in order to reduce a portion of $\Omega$ without invalidating the governing variational formulation. Then, let us assume that no longer is condition (3) satisfied, so it is necessary to rewrite Problem 1 introducing terms involving the virtual power of the jumps in $\mathbf{u}$ and in its variation $\mathbf{v}$. Thus, the extended variational principle reads as follows:

**Problem 2** *For some* $\gamma \in [0, 1]$ *find* $((\mathbf{u}_1, \mathbf{u}_2), \mathbf{t}_1, \mathbf{t}_2) \in \boldsymbol{\mathcal{U}}_d \times \boldsymbol{\mathcal{Z}}_1 \times \boldsymbol{\mathcal{Z}}_2$ *such that*

$$
\begin{aligned}
\int_{\Omega_1} \boldsymbol{\sigma}_1 \cdot \nabla \mathbf{v}_1 \, \mathrm{d}\mathbf{x} + \int_{\Omega_2} \boldsymbol{\sigma}_2 \cdot \nabla \mathbf{v}_2 \, \mathrm{d}\mathbf{x} = \\
\gamma \int_{\Gamma_a} \mathbf{t}_1 \cdot (\mathbf{v}_1 - \mathbf{v}_2) \, \mathrm{d}\Gamma + (1 - \gamma) \int_{\Gamma_a} \mathbf{t}_2 \cdot (\mathbf{v}_1 - \mathbf{v}_2) \, \mathrm{d}\Gamma \\
+ \gamma \int_{\Gamma_a} \mathbf{s}_1 \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, \mathrm{d}\Gamma + (1 - \gamma) \int_{\Gamma_a} \mathbf{s}_2 \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, \mathrm{d}\Gamma \\
+ \int_{\Omega_1} \mathbf{f} \cdot \mathbf{v}_1 \, \mathrm{d}\mathbf{x} + \int_{\Omega_2} \mathbf{f} \cdot \mathbf{v}_2 \, \mathrm{d}\mathbf{x} + \int_{\Gamma_{N_1}} \bar{\mathbf{t}}_1 \cdot \mathbf{v}_1 \, \mathrm{d}\Gamma + \int_{\Gamma_{N_2}} \bar{\mathbf{t}}_2 \cdot \mathbf{v}_2 \, \mathrm{d}\Gamma \\
\forall ((\mathbf{v}_1, \mathbf{v}_2), \mathbf{s}_1, \mathbf{s}_2) \in \boldsymbol{\mathcal{V}}_d \times \boldsymbol{\mathcal{Z}}_1 \times \boldsymbol{\mathcal{Z}}_2, \quad (5)
\end{aligned}
$$

*where* $\bar{\mathbf{t}}_1 = \bar{\mathbf{t}}_{|\Gamma_{N1}}$ *and* $\bar{\mathbf{t}}_2 = \bar{\mathbf{t}}_{|\Gamma_{N2}}$ *and also* $\boldsymbol{\mathcal{U}}_d = \boldsymbol{\mathcal{U}}_1 \times \boldsymbol{\mathcal{U}}_2$ *with*

$$
\begin{aligned}
\boldsymbol{\mathcal{U}}_1 &= \{\mathbf{u}_1 \in \mathbf{H}^1(\Omega_1);\ \mathbf{u}_{1|\Gamma_{D1}} = \bar{\mathbf{u}}_1\}, \\
\boldsymbol{\mathcal{U}}_2 &= \{\mathbf{u}_2 \in \mathbf{H}^1(\Omega_2);\ \mathbf{u}_{2|\Gamma_{D2}} = \bar{\mathbf{u}}_2\},
\end{aligned} \quad (6)
$$

*where* $\bar{\mathbf{u}}_1 = \bar{\mathbf{u}}_{|\Gamma_{D1}}$ *and* $\bar{\mathbf{u}}_2 = \bar{\mathbf{u}}_{|\Gamma_{D2}}$, $\boldsymbol{\mathcal{V}}_d = \boldsymbol{\mathcal{V}}_1 \times \boldsymbol{\mathcal{V}}_2$, *being* $\boldsymbol{\mathcal{V}}_1$ *and* $\boldsymbol{\mathcal{V}}_2$ *the spaces whose elements are differences of elements in sets* $\boldsymbol{\mathcal{U}}_1$ *and* $\boldsymbol{\mathcal{U}}_2$ *respectively. In this case it results* $\boldsymbol{\mathcal{Z}}_1 = \boldsymbol{\mathcal{Z}}_2 = \mathbf{H}^{-1/2}(\Gamma_a)$*, while the rest of the elements is defined according to Problem 1.*

It follows, in the distributional sense, that the Euler–Lagrange equations corresponding to the extended variational formulation (5) are the following

$$\begin{cases} -\operatorname{div}\boldsymbol{\sigma}_1 = \mathbf{f} & \text{in } \Omega_1, \\ -\operatorname{div}\boldsymbol{\sigma}_2 = \mathbf{f} & \text{in } \Omega_2, \\ \mathbf{u}_1 = \bar{\mathbf{u}}_1 & \text{on } \Gamma_{D1}, \\ \mathbf{u}_2 = \bar{\mathbf{u}}_2 & \text{on } \Gamma_{D2}, \\ \boldsymbol{\sigma}_1\mathbf{n}_1 = \bar{\mathbf{t}}_1 & \text{on } \Gamma_{N1}, \\ \boldsymbol{\sigma}_2\mathbf{n}_2 = \bar{\mathbf{t}}_2 & \text{on } \Gamma_{N2}, \\ \gamma(\mathbf{u}_1 - \mathbf{u}_2) = 0 & \text{on } \Gamma_a, \\ (1-\gamma)(\mathbf{u}_1 - \mathbf{u}_2) = 0 & \text{on } \Gamma_a, \\ \gamma\mathbf{t}_1 + (1-\gamma)\mathbf{t}_2 = \boldsymbol{\sigma}_1\mathbf{n}_1 & \text{on } \Gamma_a, \\ \gamma\mathbf{t}_1 + (1-\gamma)\mathbf{t}_2 = \boldsymbol{\sigma}_2\mathbf{n}_1 & \text{on } \Gamma_a. \end{cases} \tag{7}$$

Writing the Euler–Lagrange equations just in terms of the displacement field $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ it ensues that the solution does not actually depend on the real parameter $\gamma$, in fact we have

$$\begin{cases} -\operatorname{div}\boldsymbol{\sigma}_1 = f & \text{in } \Omega_1, \\ -\operatorname{div}\boldsymbol{\sigma}_2 = f & \text{in } \Omega_2, \\ \mathbf{u}_1 = \bar{\mathbf{u}}_1 & \text{on } \Gamma_{D1}, \\ \mathbf{u}_2 = \bar{\mathbf{u}}_2 & \text{on } \Gamma_{D2}, \\ \boldsymbol{\sigma}_1\mathbf{n}_1 = \bar{\mathbf{t}}_1 & \text{on } \Gamma_{N1}, \\ \boldsymbol{\sigma}_2\mathbf{n}_2 = \bar{\mathbf{t}}_2 & \text{on } \Gamma_{N2}, \\ \mathbf{u}_1 = \mathbf{u}_2 & \text{on } \Gamma_a, \\ \boldsymbol{\sigma}_1\mathbf{n}_1 = \boldsymbol{\sigma}_2\mathbf{n}_1 & \text{on } \Gamma_a. \end{cases} \tag{8}$$

We conclude then that the solution of Problem 2 satisfies the same Euler–Lagrange equations as the solution of Problem 1. Indeed, the last two expressions of (8) correspond to conditions (3)–(4).

In particular, the following result is stated in the case of small displacements and for a linear elastic materials:

**Proposition 3** *The Problem 2 under small displacements assumptions and for a linear elastic material, that is* $\boldsymbol{\sigma}_i = \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_i) = \mathbb{D}(\nabla\mathbf{u}_i)^s$ $(i = 1, 2)$*, has a unique solution* $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2) \in \boldsymbol{\mathcal{U}}_d$ *and a unique combination* $\mathbf{t}_\gamma = \gamma\mathbf{t}_1 + (1-\gamma)\mathbf{t}_2 \in \mathbf{H}^{-1/2}_{[\![\cdot]\!]}(\Gamma_a)$*.*

**PROOF.** The proof follows the main ideas used when dealing with mixed formulations (see [6]). Notice that the problem can be written as follows

$$a(\mathbf{u},\mathbf{v}) - b(\mathbf{v},\mathbf{t}_\gamma) = l(\mathbf{v}) \qquad \forall \mathbf{v} \in \boldsymbol{\mathcal{V}}_d, \tag{9}$$

$$b(\mathbf{u},\mathbf{s}_\gamma) = 0 \qquad \forall \mathbf{s}_\gamma \in \mathbf{H}^{-1/2}_{[\![\cdot]\!]}(\Gamma_a), \tag{10}$$

with $a(\cdot,\cdot) : \boldsymbol{\mathcal{W}}_d \times \boldsymbol{\mathcal{W}}_d \to \mathbb{R}$, $b(\cdot,\cdot) : \boldsymbol{\mathcal{W}}_d \times \mathbf{H}^{-1/2}_{[\![\cdot]\!]}(\Gamma_a) \to \mathbb{R}$ e $l(\cdot) : \boldsymbol{\mathcal{W}}_d \to \mathbb{R}$ defined as

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= \int_{\Omega_1} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_1)\cdot\boldsymbol{\varepsilon}(\mathbf{v}_1)\,\mathrm{d}\mathbf{x} + \int_{\Omega_2} \mathbb{D}\boldsymbol{\varepsilon}(\mathbf{u}_2)\cdot\boldsymbol{\varepsilon}(\mathbf{v}_2)\,\mathrm{d}\mathbf{x}, \\ b(\mathbf{u},\mathbf{s}_\gamma) &= \int_{\Gamma_a} (\gamma\mathbf{s}_1 + (1-\gamma)\mathbf{s}_2)\cdot(\mathbf{u}_1 - \mathbf{u}_2)\,\mathrm{d}\Gamma, \\ l(\mathbf{v}) &= \int_{\Omega_1} \mathbf{f}\cdot\mathbf{v}_1\,\mathrm{d}\mathbf{x} + \int_{\Omega_2} \mathbf{f}\cdot\mathbf{v}_2\,\mathrm{d}\mathbf{x} + \int_{\Gamma_{N_1}} \bar{\mathbf{t}}_1\cdot\mathbf{v}_1\,\mathrm{d}\Gamma + \int_{\Gamma_{N_2}} \bar{\mathbf{t}}_2\cdot\mathbf{v}_2\,\mathrm{d}\Gamma, \end{aligned} \tag{11}$$

and where $\boldsymbol{\mathcal{W}}_d = \mathbf{H}^1(\Omega_1) \times \mathbf{H}^1(\Omega_2)$ is endowed with the norm $\|\mathbf{u}\|_{\boldsymbol{\mathcal{W}}_d} = \|\mathbf{u}_1\|_{\mathbf{H}^1(\Omega_1)} + \|\mathbf{u}_2\|_{\mathbf{H}^1(\Omega_2)}$. Consider the decomposition $\mathbf{u} = \mathbf{w} + \mathbf{z}$ where $\mathbf{z} \in \boldsymbol{\mathcal{W}}_d$ is such that $\mathbf{z}_{1|\Gamma_{D1}} = \bar{\mathbf{u}}_1$, $\mathbf{z}_{1|\Gamma_a} = 0$, $\mathbf{z}_{2|\Gamma_{D2}} = \bar{\mathbf{u}}_2$ e $\mathbf{z}_{2|\Gamma_a} = 0$, while $\mathbf{w} \in \boldsymbol{\mathcal{K}} \subset \mathrm{Ker}(\mathcal{B})$ being $\mathcal{B}$ the operator associated to the form $b(\cdot,\cdot)$ and

$$\boldsymbol{\mathcal{K}} = \Big\{ \mathbf{w} = (\mathbf{w}_1,\mathbf{w}_2) \in \boldsymbol{\mathcal{W}}_d;\ \mathbf{w}_{1|\Gamma_{D1}} = 0,\ \mathbf{w}_{2|\Gamma_{D2}} = 0,\ \mathbf{w}_{1|\Gamma_a} = \mathbf{w}_{2|\Gamma_a} \Big\}. \tag{12}$$

It is quite standard to prove that the form $a(\cdot,\cdot)$ is bilinear, symmetric, continuous and coercive in $\boldsymbol{\mathcal{K}}\times\boldsymbol{\mathcal{K}}$ as $\mathbb{D}$ is a fourth–order tensor such that $\mathbb{D}\mathbf{S}\cdot\mathbf{S} > 0$, for all symmetric second–order tensor $\mathbf{S}$, and also that the form $l(\cdot)$ is linear and continuous in $\boldsymbol{\mathcal{K}}$. It follows, by using the Lax–Milgram theorem (see [6]) that there exists a unique function $\mathbf{w} \in \boldsymbol{\mathcal{K}} \subset \mathrm{Ker}(\mathcal{B})$ such that

$$a(\mathbf{w},\mathbf{v}) = l_{\mathbf{z}}(\mathbf{v}) \qquad \forall \mathbf{v} \in \boldsymbol{\mathcal{K}}, \tag{13}$$

where $l_{\mathbf{z}}(\cdot) = l(\cdot) - a(\mathbf{z},\cdot)$. Hence, the existence and uniqueness of $\mathbf{u} = (\mathbf{u}_1,\mathbf{u}_2) \in \boldsymbol{\mathcal{U}}_d$ follows. To prove the existence and uniqueness of the combination $\mathbf{t}_\gamma$ it is necessary to use the theory of mixed formulations. In this case the form $b(\cdot,\cdot)$ must satisfy an inf–sup condition so as to prove the result. Then, the space $\mathbf{H}^{-1/2}_{[\![\cdot]\!]}(\Gamma_a)$ equipped with the norm

$$\|\mathbf{t}_\gamma\|_{\mathbf{H}^{-1/2}_{[\![\cdot]\!]}(\Gamma_a)} = \sup_{\substack{[\![\mathbf{w}]\!] \in \mathbf{H}^{1/2}_{[\![\cdot]\!]}(\Gamma_a) \\ [\![\mathbf{w}]\!] \neq 0}} \frac{\displaystyle\int_{\Gamma_a} \mathbf{t}_\gamma \cdot [\![\mathbf{w}]\!]\,\mathrm{d}\Gamma}{\|[\![\mathbf{w}]\!]\|_{\mathbf{H}^{1/2}_{[\![\cdot]\!]}(\Gamma_a)}}, \tag{14}$$

is defined as the dual space of $\mathbf{H}^{1/2}_{[\![\cdot]\!]}(\Gamma_a)$ that, in turn, is defined as

$$\mathbf{H}^{1/2}_{[\![\cdot]\!]}(\Gamma_a) = \{ [\![\mathbf{w}]\!] \in \mathbf{H}^{1/2}(\Gamma_a);\ \mathbf{w} \in \boldsymbol{\mathcal{W}}_d,\ [\![\mathbf{w}]\!] = \mathbf{w}_{1|\Gamma_a} - \mathbf{w}_{2|\Gamma_a} \}, \tag{15}$$

and equipped with the norm

$$\|[\![\mathbf{w}]\!]\|_{\mathbf{H}^{1/2}_{[\![\cdot]\!]}(\Gamma_a)} = \inf_{\substack{\mathbf{y}\in\boldsymbol{\mathcal{W}}_d \\ [\![\mathbf{w}]\!]=\mathbf{y}_{1|\Gamma_a}-\mathbf{y}_{2|\Gamma_a}}} \|\mathbf{y}\|_{\boldsymbol{\mathcal{W}}_d}. \tag{16}$$

Thus, for any $\beta_1 > 1$ it is possible to take $\mathbf{z} \in \boldsymbol{\mathcal{W}}_d$ such that $\mathbf{z}_{1|\Gamma_a} - \mathbf{z}_{2|\Gamma_a} = [\![\mathbf{w}]\!]$ and

$$\|\mathbf{z}\|_{\boldsymbol{\mathcal{W}}_d} \leq \beta_1 \|[\![\mathbf{w}]\!]\|_{\mathbf{H}^{1/2}_{[\![\cdot]\!]}(\Gamma_a)}. \tag{17}$$

Using definition (14) together with expression (17) it follows that, for $\beta_2 > 1$, we have

$$\begin{aligned}\frac{1}{\beta_2}\|\mathbf{t}_\gamma\|_{\mathbf{H}^{-1/2}_{[\![\cdot]\!]}(\Gamma_a)} &< \sup_{\substack{[\![\mathbf{w}]\!]\in\mathbf{H}^{1/2}_{[\![\cdot]\!]}(\Gamma_a) \\ [\![\mathbf{w}]\!]\neq 0}} \frac{\displaystyle\int_{\Gamma_a} \mathbf{t}_\gamma \cdot [\![\mathbf{w}]\!]\, d\Gamma}{\|[\![\mathbf{w}]\!]\|_{\mathbf{H}^{1/2}_{[\![\cdot]\!]}(\Gamma_a)}} \\ &\leq \beta_1 \sup_{\substack{\mathbf{z}\in\boldsymbol{\mathcal{W}}_d \\ [\![\mathbf{w}]\!]=\mathbf{z}_{1|\Gamma_a}-\mathbf{z}_{2|\Gamma_a} \\ \mathbf{z}_{1|\Gamma_a}\neq\mathbf{z}_{2|\Gamma_a}}} \frac{\displaystyle\int_{\Gamma_a} \mathbf{t}_\gamma \cdot [\![\mathbf{w}]\!]\, d\Gamma}{\|\mathbf{z}\|_{\boldsymbol{\mathcal{W}}_d}}.\end{aligned} \tag{18}$$

Therefore, there exists $\beta_0 = \frac{1}{\beta_1\beta_2} > 0$ such that the form $b(\cdot,\cdot)$ satisfies the following inf–sup condition

$$\beta_0 \leq \inf_{\substack{\mathbf{t}_\gamma\in\mathbf{H}^{-1/2}_{[\![\cdot]\!]}(\Gamma_a) \\ \mathbf{t}_\gamma\neq 0}} \sup_{\substack{\mathbf{z}\in\boldsymbol{\mathcal{W}}_d \\ [\![\mathbf{w}]\!]=\mathbf{z}_{1|\Gamma_a}-\mathbf{z}_{2|\Gamma_a} \\ \mathbf{z}_{1|\Gamma_a}\neq\mathbf{z}_{2|\Gamma_a}}} \frac{\displaystyle\int_{\Gamma_a} \mathbf{t}_\gamma \cdot [\![\mathbf{w}]\!]\, d\Gamma}{\|\mathbf{z}\|_{\boldsymbol{\mathcal{W}}_d}\|\mathbf{t}_\gamma\|_{\mathbf{H}^{-1/2}_{[\![\cdot]\!]}(\Gamma_a)}}. \tag{19}$$

The existence and uniqueness of the combination $\mathbf{t}_\gamma \in \mathbf{H}^{-1/2}_{[\![\cdot]\!]}(\Gamma_a)$ follows. □

### *2.3 Remarks on the kinematical hypotheses*

In the preceding section the general variational setting that will be employed in the forthcoming ones was introduced. Notice that the general variational problem 2 is comprised of two kinematics, one for each component part of $\Omega$. The possibility of having two non–necessarily compatible kinematics allows us to incorporate kinematical restrictions over just a portion of the domain $\Omega$ without violating the validity of the governing principle.

The next step is to perform, within the context of variational problem 2, additional hypotheses in the governing kinematics over, say, $\Omega_1$. According to the modelling requirements it would be possible to reduce the full 3D model to a 2D, or even a 1D, model. It depends on the particular form assumed for the displacement vector field $\mathbf{u}_1$, that obviously affects the kinematical description of the admissible variations $\mathbf{v}_1$.

An important remark must be made at this point concerning the characterization of the admissible loadings. We will focus our attention on the terms defined over $\Gamma_a$. For instance, over that boundary, it was explicitly shown that the power exerted by the tractions $\mathbf{s}_1$ and $\mathbf{s}_2$ with the jump $\mathbf{u}_1 - \mathbf{u}_2$ was given by

$$\begin{aligned} \langle \mathbf{s}_1, \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\mathbf{H}^{-1/2}(\Gamma_a) \times \mathbf{H}^{1/2}(\Gamma_a)} &= \int_{\Gamma_a} \mathbf{s}_1 \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, \mathrm{d}\Gamma, \\ \langle \mathbf{s}_2, \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\mathbf{H}^{-1/2}(\Gamma_a) \times \mathbf{H}^{1/2}(\Gamma_a)} &= \int_{\Gamma_a} \mathbf{s}_2 \cdot (\mathbf{u}_1 - \mathbf{u}_2) \, \mathrm{d}\Gamma, \end{aligned} \tag{20}$$

but, actually, these expressions are the formal ones corresponding to the duality products of $\mathbf{H}^{-1/2}(\Gamma_a) \times \mathbf{H}^{1/2}(\Gamma_a)$ as a result of being working with $\boldsymbol{\mathcal{U}}_i \subset \mathbf{H}^1(\Omega_i)$, $i = 1, 2$. Now let us introduce the functional space composed by all the traces over $\Gamma_a$ of functions in $\boldsymbol{\mathcal{V}}_i$, and let us denote it by $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_i)$, $i = 1, 2$. For Problem 2 according to this notation we have

$$\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_i) = \mathcal{T}_{\Gamma_a}(\mathbf{H}^1(\Omega_i)) = \mathbf{H}^{1/2}(\Gamma_a) \qquad i = 1, 2. \tag{21}$$

Hence, the general expressions for the duality products (20) are

$$\langle \mathbf{s}_1, \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)}, \tag{22a}$$
$$\langle \mathbf{s}_2, \mathbf{u}_1 - \mathbf{u}_2 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)}, \tag{22b}$$

where $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_i)^*$ is the dual space of $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_i)$, $i = 1, 2$. Both expressions presented above in (22) cannot be explicitly given until the sets $\boldsymbol{\mathcal{U}}_1$ and $\boldsymbol{\mathcal{U}}_2$ are specified.

When performing hypotheses, for example by giving a particular form to $\mathbf{u}_1$, we automatically have, by duality arguments, that the admissible loadings (the dual elements, in this case $\mathbf{s}_1$), have a particular form according to such kinematical assumptions. This is due to the definition of set $\boldsymbol{\mathcal{U}}_1$, which goes along with the definition of space $\boldsymbol{\mathcal{V}}_1$ and hence of $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)$. Thus, as remarked in the previous paragraph, form (22a) has a different expression than the first one of (20). That is, the power exerted by $\mathbf{s}_1$ takes another form that is consistent with the kinematical assumptions made on $\mathbf{u}_1$. Whereas there are no problems in expressing the product $\langle \mathbf{s}_1, \mathbf{u}_1 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)}$, as a result of the natural duality between the elements, the question that arises now is how to consider the product $\langle \mathbf{s}_1, \mathbf{u}_2 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)}$ as the elements do not comprise a dual pair (note that $\mathbf{u}_2 \in \boldsymbol{\mathcal{U}}_2$ and $\mathbf{u}_{2|\Gamma_a} \in \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)$). To answer this question we have to recall that $\mathbf{s}_1$ is the traction vector such that together with a function of the form of $\mathbf{u}_1$ exerts a quantity of power. Therefore, any function with no components of the form of $\mathbf{u}_1$ must be orthogonal to $\mathbf{s}_1$ in the sense given by the projection defined by the duality product. That means that, being $\mathbf{u}_2$ a general function, it has to be projected in the sense of the duality product defined in $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)$ with the aim of recasting the corresponding component that resembles the form of $\mathbf{u}_1$. In order to achieve this, it is necessary to set $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)$ such that it embraces the regularity conditions that characterize $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)$. Therefore, in general we write

$$\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2) = \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1) \oplus \boldsymbol{\mathcal{W}}, \tag{23}$$

where $\boldsymbol{\mathcal{W}}$ is a general space such that in the previous decomposition of $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)$ it is orthogonal to $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^*$ in the sense given by the duality product. Thus, it is

$$\mathbf{u}_{2|\Gamma_a} = \mathbf{u}_{21} + \mathbf{u}_{2r}, \tag{24}$$

with $\mathbf{u}_{21} \in \boldsymbol{\mathcal{T}}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)$ and $\mathbf{u}_{2r} \in \boldsymbol{\mathcal{W}}$. Since $\mathbf{u}_{2r} \in \boldsymbol{\mathcal{W}} = \mathrm{Ker}(\boldsymbol{\mathcal{S}}_1)$ (the kernel of the linear functional $\boldsymbol{\mathcal{S}}_1$ associated to the element $\mathbf{s}_1$) it is possible to assert, for a general function $\mathbf{u}_2 \in \boldsymbol{\mathcal{V}}_2$, that

$$\langle \mathbf{s}_1, \mathbf{u}_2 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = \langle \mathbf{s}_1, \mathbf{u}_{21} \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)}. \tag{25}$$

From the mechanical point of view, decomposition (23) is such that functions of the form of $\mathbf{u}_{2r}$ do not exert power in duality with the elements of $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^*$. As a consequence of this, $\mathbf{u}_{2r}$ will be regarded hereafter as a fluctuation component that is *invisible* for the duality purposes according to the stated in the above. In fact, it will be seen in each example that functions in $\boldsymbol{\mathcal{W}}$ have particular properties that are strongly related to the characteristics of the kinematical assumptions considered over $\boldsymbol{\mathcal{V}}_1$.

**Remark 4** *In the general problem 2 the equivalence of the solution with respect to the real parameter $\gamma$ was firmly stated in terms of the Euler–Lagrange equations. Nevertheless, it will be seen that this property is lost when making additional hypotheses over one of the kinematics. This is a direct consequence of how the duality products are affected over $\Gamma_a$ when $\gamma$, what is manifested in the Euler–Lagrange equations, in particular in the coupling conditions. In other words, notice that the convex combination $\mathbf{s}_\gamma = \gamma\mathbf{s}_1 + (1-\gamma)\mathbf{s}_2$ is such that $\mathbf{s}_{\gamma=1} \in \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^*$ and $\mathbf{s}_{\gamma\neq 1} \in \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)^*$, that is, the space of $\mathbf{s}_\gamma$ alters its characteristics according to the value of $\gamma$.*

In the next sections two examples are addressed in order to expose how practical situations are handled using the ideas presented in the above.

## 3 Coupling 3D solid and 2D shell models

The coupling between a full 3D solid and a structural component with the characteristics of a shell is of utmost interest for diverse applications. Consider a structural component that can be considered in part as a shell. However, as a result of a rather intricate geometry or general loadings acting on the other part it has to be partially represented as a full 3D solid model. In such cases the use of coupled 3D models with 2D shell models appears as an interesting solution in order to avoid high computational cost when thinking of finding approximate solutions in a whole 3D domain.

Consider the scheme shown in Figure 2, where a general structural component can be split up into sub–domains $\Omega_1$ and $\Omega_2$ by the artificial internal boundary $\Gamma_a$.

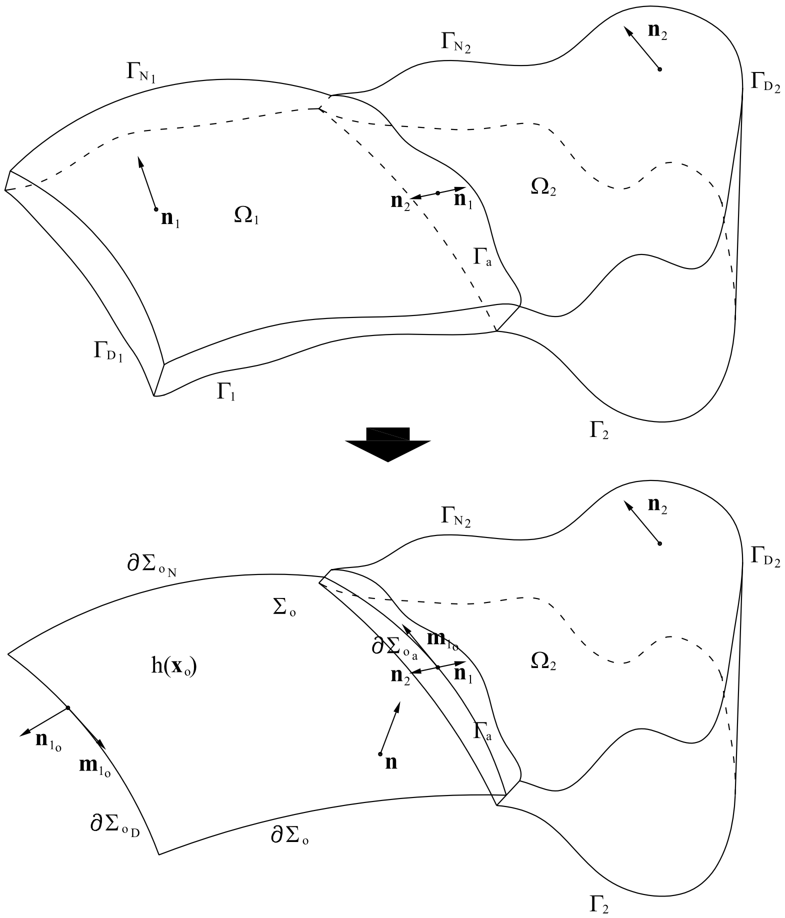


Fig. 2. Coupling 3D solid–2D shell models.

On one hand, it can be seen that the sub–domain $\Omega_2$ has an arbitrary form, and we can hardly make assumptions over the corresponding displacement field. In this part of the domain $\Omega$ we consider the full kinematics of the 3D solid model. On the other hand, the sub–domain $\Omega_1$ has a particular form that permits us to consider the analysis of this part of the component as it was a shell. In this manner a discontinuity in the displacement field arises as a consequence of the incompatible kinematics chosen for describing both portions of the component defined completely in $\Omega$.

Over the part $\Omega_1$ we carry out the decomposition of fields in terms of the tangential and normal components in order to correctly introduce the kinematical hypotheses. A fast revision about this decomposition is given in what follows.

### *3.1 Preliminaries*

A convenient manner to characterize the domain $\Omega_1$ is the following

$$\Omega_1 = \{\mathbf{x} \in \mathbb{R}^3;\ \mathbf{x} = \mathbf{x}_o + \xi \mathbf{n},\ \mathbf{x}_o \in \Sigma_o,\ \xi \in H\}, \tag{26}$$

where $\Sigma_o$ is the middle surface on which the normal vector $\mathbf{n} = \mathbf{n}(\mathbf{x}_o)$ is defined, together with the thickness of the shell $h = h(\mathbf{x}_o)$ given by the interval $H = (-\frac{h(\mathbf{x}_o)}{2}, \frac{h(\mathbf{x}_o)}{2})$. The relation $\mathbf{x} \leftrightarrow (\mathbf{x}_o, \xi)$ is uniquely determined provided the thickness of the shell is less than twice the largest radius of curvature. Domain $\Omega_1$ is limited from top and bottom by surfaces $\Sigma^+$ and $\Sigma^-$ respectively, and is laterally limited by surface $\Gamma_L$. In particular we identify the lateral coupling surface $\Gamma_a$. With this description in mind, we also assume that the shell is smooth, so we have a unique normal and tangent plane over each point $\mathbf{x}_o \in \Sigma_o$.

To conveniently express the virtual power principle in the case of a shell, a decomposition of tensors $\boldsymbol{\sigma}_1$ and $\nabla \mathbf{v}_1$ in terms of the tangential and normal components should be given. Firstly, by making use of the projection operator over the tangent plane, denoted by $\mathbf{\Pi}_t(\mathbf{x}_o) = \mathbf{I} - \mathbf{n}(\mathbf{x}_o) \otimes \mathbf{n}(\mathbf{x}_o)$, it is easy to see that, for a symmetric tensor $\mathbf{S}$ we have the following decomposition

$$\mathbf{S} = \mathbf{S}_t + \mathbf{S}_s \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{S}_s + S_n(\mathbf{n} \otimes \mathbf{n}), \tag{27}$$

where $\mathbf{S}_t = \mathbf{\Pi}_t \mathbf{S} \mathbf{\Pi}_t$ (tensor), $\mathbf{S}_s = \mathbf{\Pi}_t \mathbf{S} \mathbf{n}$ (vector) and $S_n = (\mathbf{S}\mathbf{n}) \cdot \mathbf{n}$ (scalar). Recall that this is an orthogonal decomposition. Now, consider the vector field $\mathbf{v}$ written in tangential and normal components, that is $\mathbf{v} = \mathbf{v}_t + v_n \mathbf{n}$. In this way, by simply differentiating yields the following

$$\nabla \mathbf{v} = (\nabla \mathbf{v})_t + (\nabla \mathbf{v})_s \otimes \mathbf{n} + \mathbf{n} \otimes (\nabla \mathbf{v})_s^* + (\nabla \mathbf{v})_n (\mathbf{n} \otimes \mathbf{n}), \tag{28}$$

where

$$\begin{aligned}
(\nabla \mathbf{v})_t &= \mathbf{\Pi}_t (\nabla_{\mathbf{x}_o} \mathbf{v}_t) \mathbf{\Lambda}^{-1} \mathbf{\Pi}_t + v_n (\nabla_{\mathbf{x}_o} \mathbf{n}) \mathbf{\Lambda}^{-1} \mathbf{\Pi}_t, \\
(\nabla \mathbf{v})_s &= \frac{\partial \mathbf{v}_t}{\partial \xi}, \\
(\nabla \mathbf{v})_s^* &= \mathbf{\Pi}_t \mathbf{\Lambda}^{-1} \nabla_{\mathbf{x}_o} v_n - \mathbf{\Pi}_t \mathbf{\Lambda}^{-1} (\nabla_{\mathbf{x}_o} \mathbf{n}) \mathbf{v}_t, \\
(\nabla \mathbf{v})_n &= \frac{\partial v_n}{\partial \xi},
\end{aligned} \tag{29}$$

with the invertible operator $\mathbf{\Lambda}$ defined as $\mathbf{\Lambda} = \mathbf{I} + \xi \nabla_{\mathbf{x}_o} \mathbf{n}$, where $\nabla_{\mathbf{x}_o}(\cdot)$ denotes the gradient in the variable $\mathbf{x}_o$ defined over $\Sigma_o$ and $\nabla_{\mathbf{x}_o} \mathbf{n}$ is the curvature tensor over the surface $\Sigma_o$.

Observe that the following equivalent ways of writing the integrals will be useful to state the problem over $\Omega_1$

$$
\begin{aligned}
\int_{\Omega_1} (\cdot)\, \mathrm{d}\mathbf{x} &= \int_{\Sigma_o} \int_H (\cdot) \det \mathbf{\Lambda}\, \mathrm{d}\xi\, \mathrm{d}\Sigma_o, \\
\int_{\Sigma^+} (\cdot)\, \mathrm{d}\Gamma &= \int_{\Sigma_o} (\cdot) \det \mathbf{\Lambda}^+\, \mathrm{d}\Sigma_o, \\
\int_{\Sigma^-} (\cdot)\, \mathrm{d}\Gamma &= \int_{\Sigma_o} (\cdot) \det \mathbf{\Lambda}^-\, \mathrm{d}\Sigma_o, \\
\int_{\Gamma_{LN}} (\cdot)\, \mathrm{d}\Gamma &= \int_{\partial\Sigma_{oN}} \int_H (\cdot) [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2}\, \mathrm{d}\xi \mathrm{d}\partial\Sigma_o, \\
\int_{\Gamma_a} (\cdot)\, \mathrm{d}\Gamma &= \int_{\partial\Sigma_{oa}} \int_H (\cdot) [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2}\, \mathrm{d}\xi \mathrm{d}\partial\Sigma_o,
\end{aligned}
\tag{30}
$$

with $\mathbf{m}_{1o} = \mathbf{n} \times \mathbf{n}_{1o}$ the unit tangent vector to $\Sigma_o$ over $\partial\Sigma_o$, being $\mathbf{n}_{1o}$ the unit outward normal to the lateral boundary $\Gamma_L$.

### *3.2 Kinematical assumptions*

The shell model in this section is established assuming that the displacement over $\Omega_1$ has the following particular form

$$
\begin{aligned}
\mathbf{u}_1(\mathbf{x}) &= \mathbf{u}_{1t}(\mathbf{x}_o, \xi) + \mathbf{u}_{1n}(\mathbf{x}_o), \\
\mathbf{u}_{1t}(\mathbf{x}_o, \xi) &= \mathbf{u}_{1t}^o(\mathbf{x}_o) + \xi \boldsymbol{\omega}_{1t}(\mathbf{x}_o), \\
\mathbf{u}_{1n}(\mathbf{x}_o) &= u_{1n}(\mathbf{x}_o) \mathbf{n}(\mathbf{x}_o),
\end{aligned}
\tag{31}
$$

where $\mathbf{u}_{1t}$ is a vector that lies on the tangent plane to the middle surface and $\mathbf{u}_{1n}$ is the component along the normal direction. For this model the normal fibers remain normal after the deformation process, and also they remain equally sized. This model is known as the Naghdi model [11,12]. It is worthwhile to mention that other models as the Reissner–Mindlin or the Kirchhoff–Love models may be equally handled, as they constitute particular cases of the situation analyzed in this section. On the other hand, over $\Omega_2$ the complete kinematics is regarded.

Once the kinematics over $\Omega_1$ was defined we are in position to characterize the form of the duality product. Firstly, notice that according to the definition (31), $\mathbf{u}_1$ is characterized by the triple $(\mathbf{u}_{1t}^o, \boldsymbol{\omega}_{1t}, u_{1n}) \in \boldsymbol{\mathcal{V}}_1$ where

$$
\boldsymbol{\mathcal{V}}_1 = \mathbf{H}^1(\Sigma_o) \times \mathbf{H}^1(\Sigma_o) \times H^1(\Sigma_o). \tag{32}
$$

Then, using duality arguments we have that any admissible traction $\mathbf{s}_1$ compatible with the kinematical assumptions (31) is such that the power that it exerts over $\Gamma_a$ is expressed as

$$\langle \mathbf{s}_1, \mathbf{u}_1 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = \langle (\mathbf{s}_1{}_t^o, \boldsymbol{\nu}_{1t}, s_{1n}), (\mathbf{u}_1{}_t^o, \boldsymbol{\omega}_{1t}, u_{1n}) \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = \int_{\partial\Sigma_{oa}} \left[ \mathbf{s}_1{}_t^o \cdot \mathbf{u}_1{}_t^o + \boldsymbol{\nu}_{1t} \cdot \boldsymbol{\omega}_{1t} + s_{1n} u_{1n} \right] \mathrm{d}\partial\Sigma_o, \quad (33)$$

where $\partial\Sigma_{oa}$ is the corresponding part of $\Gamma_a$ over the middle surface of the shell. We easily identify that the traction compatible with the kinematics has the form of a triple $\mathbf{s}_1 = (\mathbf{s}_1{}_t^o, \boldsymbol{\nu}_{1t}, s_{1n})$, being $\mathbf{s}_1{}_t^o$, $\boldsymbol{\nu}_{1t}$ and $s_{1n}$ the dual elements of $\mathbf{u}_1{}_t^o$, $\boldsymbol{\omega}_{1t}$ and $u_{1n}$ respectively, and that also they depend only on $\mathbf{x}_o$. Here it is easy to recognize that these generalized loadings correspond to membrane, flexion and normal effects, respectively. For this case we have that $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)$ and $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^*$ are given by

$$\begin{aligned} \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1) &= \mathbf{H}^{1/2}(\partial\Sigma_{oa}) \times \mathbf{H}^{1/2}(\partial\Sigma_{oa}) \times H^{1/2}(\partial\Sigma_{oa}), \\ \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* &= \mathbf{H}^{-1/2}(\partial\Sigma_{oa}) \times \mathbf{H}^{-1/2}(\partial\Sigma_{oa}) \times H^{-1/2}(\partial\Sigma_{oa}). \end{aligned} \quad (34)$$

Concerning what was commented in Section 2.3, the question is how to handle a product of the form $\langle \mathbf{s}_1, \mathbf{u}_2 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)}$. According to the ideas developed in that section observe that the space $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)$ has to be such that it can be written as in expression (23) with $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)$ given by (34). The need for performing the projection of $\mathbf{u}_2$ in the sense given by the duality product (33) obliges $\boldsymbol{\mathcal{U}}_2$ to be such that the decomposition of an arbitrary $\mathbf{u}_{2|\Gamma_a}$ as done in (24) holds, for which $\mathbf{u}_{21}$ is such that

$$\mathbf{u}_{21}(\mathbf{x}_o, \xi) = \mathbf{u}_2{}_t^o(\mathbf{x}_o) + \xi \boldsymbol{\omega}_2{}_t^o(\mathbf{x}_o) + u_2{}_n^o(\mathbf{x}_o)\mathbf{n}(\mathbf{x}_o) \qquad \forall (\mathbf{x}_o, \xi) \in \Gamma_a, \quad (35)$$

where the decomposition into tangent and normal components was employed. Consequently, we have now that the projection of $\mathbf{u}_2 \in \boldsymbol{\mathcal{U}}_2$ in the sense of the operation $\langle \cdot, \cdot \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)}$ is given by

$$\langle \mathbf{s}_1, \mathbf{u}_2 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = \langle \mathbf{s}_1, \mathbf{u}_{21} \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = \langle (\mathbf{s}_1{}_t^o, \boldsymbol{\nu}_{1t}, s_{1n}), (\mathbf{u}_2{}_t^o, \boldsymbol{\omega}_2{}_t^o, u_2{}_n^o) \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = \int_{\partial\Sigma_{oa}} \left[ \mathbf{s}_1{}_t^o \cdot \mathbf{u}_2{}_t^o + \boldsymbol{\nu}_{1t} \cdot \boldsymbol{\omega}_2{}_t^o + s_{1n} u_2{}_n^o \right] \mathrm{d}\partial\Sigma_o, \quad (36)$$

as the fluctuation component $\mathbf{u}_{2r}$ is such that

$$\langle \mathbf{s}_1, \mathbf{u}_{2r} \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = 0 \qquad \forall \mathbf{s}_1 \in \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^*. \quad (37)$$

**Remark 5** *A consequence of having introduced hypothesis* (31) *is that the regularity of set* $\boldsymbol{\mathcal{U}}_2$ *is no longer given by space* $\mathbf{H}^1(\Omega_2)$*, since additional regularity is needed in order to be able to perform the projection of* $\mathbf{u}_2$ *given by the duality product.*

From (37) it is not difficult to see that the fluctuations $\mathbf{u}_{2r}$ have the following additional properties

$$
\begin{aligned}
\mathbf{\Pi}_t \int_H \mathbf{u}_{2r} [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \, \mathrm{d}\xi &= 0, \\
\mathbf{\Pi}_t \int_H \frac{\partial \mathbf{u}_{2r}}{\partial \xi} [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \, \mathrm{d}\xi &= 0, \\
\int_H \mathbf{u}_{2r} \cdot \mathbf{n} [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \, \mathrm{d}\xi &= 0,
\end{aligned}
\tag{38}
$$

contrariwise no longer is expression (37) valid neither does the orthogonal decomposition (23) hold for this case. Consequently, from properties (38) we obtain that

$$
\begin{aligned}
H_{\xi^0} \mathbf{u}_{2t}^o + H_{\xi^1} \boldsymbol{\omega}_{2t}^o &= \mathbf{\Pi}_t \int_H \mathbf{u}_2 [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \, \mathrm{d}\xi, \\
H_{\xi^0} \boldsymbol{\omega}_{2t}^o &= \mathbf{\Pi}_t \int_H \frac{\partial \mathbf{u}_2}{\partial \xi} [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \, \mathrm{d}\xi, \\
H_{\xi^0} u_{2n}^o &= \int_H \mathbf{u}_2 \cdot \mathbf{n} [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \, \mathrm{d}\xi,
\end{aligned}
\tag{39}
$$

where $H_{\xi^i} = \int_H \xi^i [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}] \, \mathrm{d}\xi$. Then, decomposition (24) is completely defined with $\mathbf{u}_{2r}$ satisfying (37), while $\mathbf{u}_{21}$ is given by (35). Furthermore, from (39) $\mathbf{u}_{21}$ can be characterized as follows

$$
\begin{aligned}
\mathbf{u}_{2t}^o &= \frac{1}{H_{\xi^0}} \mathbf{\Pi}_t \int_H \left( \mathbf{u}_2 - \frac{H_{\xi^1}}{H_{\xi^0}} \frac{\partial \mathbf{u}_2}{\partial \xi} \right) [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \, \mathrm{d}\xi, \\
\boldsymbol{\omega}_{2t}^o &= \frac{1}{H_{\xi^0}} \mathbf{\Pi}_t \int_H \frac{\partial \mathbf{u}_2}{\partial \xi} [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \, \mathrm{d}\xi, \\
u_{2n}^o &= \frac{1}{H_{\xi^0}} \int_H \mathbf{u}_2 \cdot \mathbf{n} [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \, \mathrm{d}\xi.
\end{aligned}
\tag{40}
$$

Now, we are in position to establish the regularity requirements of functions in $\Omega_2$

$$
\begin{aligned}
\boldsymbol{\mathcal{V}}_2 = \{ \mathbf{v}_2 \in \mathbf{H}^1(\Omega_2);\ & \mathbf{v}_{2|\Gamma_a} = \mathbf{v}_{2t}^o + \xi \boldsymbol{\varphi}_{2t}^o + v_{2n}^o \mathbf{n} + \mathbf{v}_{2r}; \\
& (\mathbf{v}_{2t}^o, \boldsymbol{\varphi}_{2t}^o, v_{2n}^o) \in \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1);\ \mathbf{v}_{2r} \text{ satisfies (38)} \}.
\end{aligned}
\tag{41}
$$

The counterpart expression for the duality product between the admissible tractions $\mathbf{s}_2$ and either the displacement field $\mathbf{u}_2$ or $\mathbf{u}_1$ is handled as usual, and given by

$$
\langle \mathbf{s}_2, \mathbf{u}_2 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)} = \int_{\Gamma_a} \mathbf{s}_2 \cdot \mathbf{u}_2 \, \mathrm{d}\Gamma. \tag{42}
$$

In what follows the decomposition of the fields defined over $\Omega_1$ in terms of the tangent and normal components with respect to the middle surface of the shell is employed. The next step consists in introducing the kinematical hypothesis (31) alongside with the duality product defined by (33), its consequence (36), and (42). Therefore, putting all this together into the variational formulation (5) leads, after a little basic work, to the following

$$\begin{aligned}
&\int_{\Sigma_o}\int_H \Big[\boldsymbol{\sigma}_{1t}\cdot(\boldsymbol{\Pi}_t\nabla_{\mathbf{x}_o}\mathbf{v}_{1t}^{o}+\xi\boldsymbol{\Pi}_t\nabla_{\mathbf{x}_o}\boldsymbol{\varphi}_{1t}+v_{1n}\nabla_{\mathbf{x}_o}\mathbf{n})\boldsymbol{\Lambda}^{-1}\\
&+\boldsymbol{\sigma}_{1s}\cdot\boldsymbol{\Lambda}^{-1}(\boldsymbol{\varphi}_{1t}-(\nabla_{\mathbf{x}_o}\mathbf{n})\mathbf{v}_{1t}^{o}+\nabla_{\mathbf{x}_o}v_{1n})\Big]\det\boldsymbol{\Lambda}\,\mathrm{d}\xi\mathrm{d}\Sigma_o+\int_{\Omega_2}\boldsymbol{\sigma}_2\cdot\nabla\mathbf{v}_2\,\mathrm{d}\mathbf{x}=\\
&\gamma\int_{\partial\Sigma_{oa}}\Big[\mathbf{t}_{1t}^{o}\cdot(\mathbf{v}_{1t}^{o}-\mathbf{v}_{2t}^{o})+\boldsymbol{\mu}_{1t}\cdot(\boldsymbol{\varphi}_{1t}-\boldsymbol{\varphi}_{2t}^{o})+t_{1n}(v_{1n}-v_{2n}^{o})\Big]\mathrm{d}\partial\Sigma_o\\
&+(1-\gamma)\int_{\Gamma_a}\mathbf{t}_2\cdot((\mathbf{v}_{1t}^{o}+\xi\boldsymbol{\varphi}_{1t}+v_{1n}\mathbf{n})-(\mathbf{v}_{2t}^{o}+\xi\boldsymbol{\varphi}_{2t}^{o}+v_{2n}^{o}\mathbf{n}+\mathbf{v}_{2r}))\,\mathrm{d}\Gamma\\
&+\gamma\int_{\partial\Sigma_{oa}}\mathbf{s}_{1t}^{o}\cdot(\mathbf{u}_{1t}^{o}-\mathbf{u}_{2t}^{o})\,\mathrm{d}\partial\Sigma_{oa}+\gamma\int_{\partial\Sigma_{oa}}\boldsymbol{\nu}_{1t}\cdot(\boldsymbol{\omega}_{1t}-\boldsymbol{\omega}_{2t}^{o})\,\mathrm{d}\partial\Sigma_{oa}\\
&+\gamma\int_{\partial\Sigma_{oa}}s_{1n}(u_{1n}-u_{2n}^{o})\,\mathrm{d}\partial\Sigma_{oa}+(1-\gamma)\int_{\Gamma_a}\mathbf{s}_2\cdot((\mathbf{u}_{1t}^{o}+\xi\boldsymbol{\omega}_{1t}+u_{1n}\mathbf{n})-\mathbf{u}_2)\,\mathrm{d}\Gamma\\
&+\int_{\Sigma_o}\int_H[\mathbf{f}_t\cdot\mathbf{v}_{1t}^{o}+\mathbf{f}_t\cdot\xi\boldsymbol{\varphi}_{1t}+f_nv_{1n}]\det\boldsymbol{\Lambda}\,\mathrm{d}\xi\,\mathrm{d}\Sigma_o+\int_{\Omega_2}\mathbf{f}\cdot\mathbf{v}_2\,\mathrm{d}\mathbf{x}\\
&+\int_{\Sigma_o}\left[\bar{\mathbf{t}}_{1t}^{+}\cdot\mathbf{v}_{1t}^{o}+\bar{\mathbf{t}}_{1t}^{+}\cdot\frac{h}{2}\boldsymbol{\varphi}_{1t}+\bar{t}_{1n}^{+}v_{1n}\right]\det\boldsymbol{\Lambda}^{+}\,\mathrm{d}\Sigma_o\\
&+\int_{\Sigma_o}\left[\bar{\mathbf{t}}_{1t}^{-}\cdot\mathbf{v}_{1t}^{o}-\bar{\mathbf{t}}_{1t}^{-}\cdot\frac{h}{2}\boldsymbol{\varphi}_{1t}+\bar{t}_{1n}^{-}v_{1n}\right]\det\boldsymbol{\Lambda}^{-}\,\mathrm{d}\Sigma_o\\
&+\int_{\partial\Sigma_{oN}}\int_H\Big[\bar{\mathbf{t}}_{1t}\cdot\mathbf{v}_{1t}^{o}+\bar{\mathbf{t}}_{1t}\cdot\xi\boldsymbol{\varphi}_{1t}+\bar{t}_{1n}v_{1n}\Big][\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi\mathrm{d}\partial\Sigma_o\\
&+\int_{\Gamma_{N_2}}\bar{\mathbf{t}}_2\cdot\mathbf{v}_2\,\mathrm{d}\Gamma\\
&\forall((\mathbf{v}_{1t}^{o}.\boldsymbol{\varphi}_{1t},v_{1n},\mathbf{v}_2),(\mathbf{s}_{1t}^{o},\boldsymbol{\nu}_{1t},s_{1n}),\mathbf{s}_2)\in\boldsymbol{\mathcal{V}}_d\times\boldsymbol{\mathcal{Z}}_1\times\boldsymbol{\mathcal{Z}}_2,\qquad(43)
\end{aligned}$$

where the spaces $\boldsymbol{\mathcal{V}}_d$, $\boldsymbol{\mathcal{Z}}_1$ e $\boldsymbol{\mathcal{Z}}_2$ are specified below. Also, the following additional decompositions and notations were considered $\mathbf{f} = \mathbf{f}_t + f_n\mathbf{n}$ in $\Omega_1$, $\bar{\mathbf{t}}_1 = \bar{\mathbf{t}}_{1t}^{+} + \bar{t}_{1n}^{+}\mathbf{n}$ on $\Sigma^+$, $\bar{\mathbf{t}}_1 = \bar{\mathbf{t}}_{1t}^{-} + \bar{t}_{1n}^{-}\mathbf{n}$ on $\Sigma^-$ and $\bar{\mathbf{t}}_1 = \bar{\mathbf{t}}_{1t} + \bar{t}_{1n}\mathbf{n}$ on $\Gamma_{LN}$. Here, index $\pm$ indicates that the quantity is evaluated at $\xi = \pm\frac{h}{2}$. Besides, boundaries $\Sigma^+$ and $\Sigma^-$ are Neumann boundaries as well as $\Gamma_{LN}$, such that we get $\Gamma_{N1} = \Sigma^+ \cup \Sigma^- \cup \Gamma_{LN}$. Recall that $\boldsymbol{\sigma}_{1t}$ is a tensor and $\boldsymbol{\sigma}_{1s}$ is a vector.

As usual in shell theory, from the above expression the following generalized stresses and loadings are defined

$$
\begin{aligned}
\mathbf{N}_{1t}^{H} &= \int_H \boldsymbol{\sigma}_{1t}\boldsymbol{\Lambda}^{-1}\det\boldsymbol{\Lambda}\,\mathrm{d}\xi,\\
\mathbf{M}_{1t}^{H} &= \int_H \boldsymbol{\sigma}_{1t}\boldsymbol{\Lambda}^{-1}\xi\det\boldsymbol{\Lambda}\,\mathrm{d}\xi,\\
\mathbf{Q}_{1t}^{H} &= \int_H \boldsymbol{\Lambda}^{-1}\boldsymbol{\sigma}_{1s}\det\boldsymbol{\Lambda}\,\mathrm{d}\xi,\\
\mathbf{f}_{t}^{H} &= \int_H \mathbf{f}_t\det\boldsymbol{\Lambda}\,\mathrm{d}\xi + \bar{\mathbf{t}}_{1t}^{+}\det\boldsymbol{\Lambda}^{+} + \bar{\mathbf{t}}_{1t}^{-}\det\boldsymbol{\Lambda}^{-},\\
f_{n}^{H} &= \int_H f_n\det\boldsymbol{\Lambda}\,\mathrm{d}\xi + \bar{t}_{1n}^{+}\det\boldsymbol{\Lambda}^{+} + \bar{t}_{1n}^{-}\det\boldsymbol{\Lambda}^{-},\\
\mathbf{m}_{t}^{H} &= \int_H \mathbf{f}_t\xi\det\boldsymbol{\Lambda}\,\mathrm{d}\xi + \frac{h}{2}\left(\bar{\mathbf{t}}_{1t}^{+}\det\boldsymbol{\Lambda}^{+} - \bar{\mathbf{t}}_{1t}^{-}\det\boldsymbol{\Lambda}^{-}\right),\\
\bar{\mathbf{t}}_{1t}^{H} &= \int_H \bar{\mathbf{t}}_{1t}[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi,\\
\bar{t}_{1n}^{H} &= \int_H \bar{t}_{1n}[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi,\\
\bar{\mathbf{m}}_{1t}^{H} &= \int_H \bar{\mathbf{t}}_{1t}\xi[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi,
\end{aligned}
\tag{44}
$$

where, summarizing, $\mathbf{N}_{1t}^{H}$ is the membrane stress tensor, $\mathbf{M}_{1t}^{H}$ is the flexion stress tensor and $\mathbf{Q}_{1t}^{H}$ is the shear stress vector. Introducing now these definitions into expression (43) we arrive at the formulation that handles the coupling of a full 3D solid model with a 2D shell model. For the sake of completeness the problem is enunciated here with all the elements involved:

**Problem 6** *For some* $\gamma \in [0,1]$ *find* $((\mathbf{u}_{1t}^{o}.\boldsymbol{\omega}_{1t}, u_{1n}, \mathbf{u}_2), (\mathbf{t}_{1t}^{o}, \boldsymbol{\mu}_{1t}, t_{1n}), \mathbf{t}_2) \in$

$\boldsymbol{\mathcal{U}}_d \times \boldsymbol{\mathcal{Z}}_1 \times \boldsymbol{\mathcal{Z}}_2$ *such that*

$$\begin{aligned}
&\int_{\Sigma_o} \Big[ {\mathbf{N}_1}_t^H \cdot (\mathbf{\Pi}_t \nabla_{\mathbf{x}_o} {\mathbf{v}_1}_t^o + v_{1n} \nabla_{\mathbf{x}_o} \mathbf{n}) + {\mathbf{M}_1}_t^H \cdot \mathbf{\Pi}_t \nabla_{\mathbf{x}_o} \boldsymbol{\varphi}_{1t} \\
&\qquad + {\mathbf{Q}_1}_t^H \cdot (\boldsymbol{\varphi}_{1t} - (\nabla_{\mathbf{x}_o} \mathbf{n}) {\mathbf{v}_1}_t^o + \nabla_{\mathbf{x}_o} v_{1n}) \Big] \,\mathrm{d}\Sigma_o + \int_{\Omega_2} \boldsymbol{\sigma}_2 \cdot \nabla \mathbf{v}_2 \,\mathrm{d}\mathbf{x} = \\
&\int_{\partial\Sigma_{oa}} \Big( \gamma {\mathbf{t}_1}_t^o + (1-\gamma) \mathbf{\Pi}_t \int_H \mathbf{t}_2 [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \,\mathrm{d}\xi \Big) \cdot ({\mathbf{v}_1}_t^o - {\mathbf{v}_2}_t^o) \,\mathrm{d}\partial\Sigma_o \\
&+ \int_{\partial\Sigma_{oa}} \Big( \gamma \boldsymbol{\mu}_{1t} + (1-\gamma) \mathbf{\Pi}_t \int_H \mathbf{t}_2 \xi [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \,\mathrm{d}\xi \Big) \cdot (\boldsymbol{\varphi}_{1t} - {\boldsymbol{\varphi}_2}_t^o) \,\mathrm{d}\partial\Sigma_o \\
&+ \int_{\partial\Sigma_{oa}} \Big( \gamma t_{1n} + (1-\gamma) \int_H \mathbf{t}_2 \cdot \mathbf{n} [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \,\mathrm{d}\xi \Big) (v_{1n} - {v_2}_n^o) \,\mathrm{d}\partial\Sigma_o \\
&\qquad - (1-\gamma) \int_{\Gamma_a} \mathbf{t}_2 \cdot \mathbf{v}_{2r} \,\mathrm{d}\Gamma \\
&\qquad + \gamma \int_{\partial\Sigma_{oa}} {\mathbf{s}_1}_t^o \cdot ({\mathbf{u}_1}_t^o - {\mathbf{u}_2}_t^o) \,\mathrm{d}\partial\Sigma_o + \gamma \int_{\partial\Sigma_{oa}} \boldsymbol{\nu}_{1t} \cdot (\boldsymbol{\omega}_{1t} - {\boldsymbol{\omega}_2}_t^o) \,\mathrm{d}\partial\Sigma_o \\
&+ \gamma \int_{\partial\Sigma_{oa}} s_{1n} (u_{1n} - {u_2}_n^o) \,\mathrm{d}\partial\Sigma_o + (1-\gamma) \int_{\Gamma_a} \mathbf{s}_2 \cdot (({\mathbf{u}_1}_t^o + \xi \boldsymbol{\omega}_{1t} + u_{1n} \mathbf{n}) - \mathbf{u}_2) \,\mathrm{d}\Gamma \\
&\qquad + \int_{\Sigma_o} \Big[ \mathbf{f}_t^H \cdot {\mathbf{v}_1}_t^o + \mathbf{m}_t^H \cdot \boldsymbol{\varphi}_{1t} + f_n^H v_{1n} \Big] \,\mathrm{d}\Sigma_o + \int_{\Omega_2} \mathbf{f} \cdot \mathbf{v}_2 \,\mathrm{d}\mathbf{x} \\
&\qquad + \int_{\partial\Sigma_{oN}} \Big[ {\bar{\mathbf{t}}_1}{}_t^H \cdot {\mathbf{v}_1}_t^o + {\bar{\mathbf{m}}_1}{}_t^H \cdot \boldsymbol{\varphi}_{1t} + {\bar{t}_1}{}_n^H v_{1n} \Big] \,\mathrm{d}\partial\Sigma_o + \int_{\Gamma_{N_2}} \bar{\mathbf{t}}_2 \cdot \mathbf{v}_2 \,\mathrm{d}\Gamma \\
&\qquad\qquad \forall (({\mathbf{v}_1}_t^o . \boldsymbol{\varphi}_{1t}, v_{1n}, \mathbf{v}_2), ({\mathbf{s}_1}_t^o, \boldsymbol{\nu}_{1t}, s_{1n}), \mathbf{s}_2) \in \boldsymbol{\mathcal{V}}_d \times \boldsymbol{\mathcal{Z}}_1 \times \boldsymbol{\mathcal{Z}}_2, \quad (45)
\end{aligned}$$

*where* $\boldsymbol{\mathcal{U}}_d = \boldsymbol{\mathcal{U}}_1 \times \boldsymbol{\mathcal{U}}_2$ *with*

$$\begin{aligned}
&\boldsymbol{\mathcal{U}}_1 = \{({\mathbf{u}_1}_t^o, \boldsymbol{\omega}_{1t}, u_{1n}) \in \boldsymbol{\mathcal{V}}_1;\ ({\mathbf{u}_1}_t^o, \boldsymbol{\omega}_{1t}, u_{1n})_{|\Gamma_{D1}} = ({\bar{\mathbf{u}}_1}{}_t^o, \bar{\boldsymbol{\omega}}_{1t}, \bar{u}_{1n})\}, \\
&\boldsymbol{\mathcal{U}}_2 = \{\mathbf{v}_2 \in \boldsymbol{\mathcal{V}}_2;\ \mathbf{v}_{2|\Gamma_{D2}} = \bar{\mathbf{v}}_2\},
\end{aligned} \qquad (46)$$

*being* $\boldsymbol{\mathcal{V}}_1$ *and* $\boldsymbol{\mathcal{V}}_2$ *defined as in* (32) *and* (41) *respectively. Also,* $\boldsymbol{\mathcal{V}}_d$ *is the associated linear space obtained from difference of elements of* $\boldsymbol{\mathcal{U}}_d$*. Likewise, we have that* $\boldsymbol{\mathcal{Z}}_1 = \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^*$ *as stated in* (34)*, whereas* $\boldsymbol{\mathcal{Z}}_2 = \mathbf{H}^{-1/2}(\Gamma_a)$*. All the other elements are defined as in Problem 2.*

Recall that the problem is correctly posed once the constitutive behaviors for the generalized stresses ${\mathbf{N}_1}_t^H$, ${\mathbf{M}_1}_t^H$ and ${\mathbf{Q}_1}_t^H$ are given as functions of fields ${\mathbf{u}_1}_t^o$, $\boldsymbol{\omega}_{1t}$ and $u_{1n}$, whilst the stress tensor $\boldsymbol{\sigma}_2$ have to be given in terms of $\mathbf{u}_2$.

**Remark 7** *Giving continuation to that mentioned in remark 5, Problem 6 demands more regularity for functions in* $\Omega_2$ *in order to have the expression* $\mathbf{\Pi}_t \int_H \frac{\partial \mathbf{u}_2}{\partial \xi} [\mathbf{\Lambda}^2 \mathbf{m}_{1o} \cdot \mathbf{m}_{1o}]^{1/2} \,\mathrm{d}\xi \in \mathbf{H}^{1/2}(\partial\Sigma_{oa})$ *well defined. This outcome is a direct consequence of the kinematics chosen over* $\Omega_1$*, where this regularity requirement is intrinsically presumed through stating expression* (31)*. This reflects that the assumptions taken over a portion of* $\Omega$ *effectively affect the problem over the complementary portion of the domain. This is a cost that must be necessarily paid in order to have well posed the corresponding duality products.*

**Remark 8** *It is worth noticing that the parameter $\gamma$ determines the sense in which the dual products are regarded in Problem 6. Certainly, revisiting expression* (45) *allows us to see that when choosing $\gamma = 1$ we are imposing that the jumps are regarded solely by the sense of the dual product $\langle\cdot,\cdot\rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^*\times\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)}$. Conversely, when $\gamma = 0$ we are including just the term $\langle\cdot,\cdot\rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)^*\times\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)}$. This aspect of the formulation will establish important features of the corresponding Euler–Lagrange equations, impacting on the sense in which the continuity of the quantities is achieved, as will be seen in what follows. Actually, the difference between both senses is given by the space $\boldsymbol{\mathcal{W}}$ of decomposition* (23)*, and by the functions, regarded as fluctuations, that pertain to that component of $\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)$.*

### *3.4 Euler–Lagrange equations*

Before proceeding to obtain the Euler–Lagrange equations, the tensor $\boldsymbol{\sigma}_2$ over $\Gamma_a$ is written, without loss of generality, in a convenient manner as follows

$$\boldsymbol{\sigma}_{2|\Gamma_a} = \bar{\boldsymbol{\sigma}}_2 + \tilde{\boldsymbol{\sigma}}_2, \tag{47}$$

where $\bar{\boldsymbol{\sigma}}_2$ and $\tilde{\boldsymbol{\sigma}}_2$ are such that

$$\begin{aligned}
&\boldsymbol{\Pi}_t \int_H \boldsymbol{\sigma}_2\mathbf{n}_{1o}[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi = H_{\xi^0}\boldsymbol{\Pi}_t\bar{\boldsymbol{\sigma}}_2\mathbf{n}_{1o},\\
&\int_H \boldsymbol{\sigma}_2\mathbf{n}_{1o}\cdot\mathbf{n}[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi = H_{\xi^0}\bar{\boldsymbol{\sigma}}_2\mathbf{n}_{1o}\cdot\mathbf{n},\\
&\boldsymbol{\Pi}_t \int_H \tilde{\boldsymbol{\sigma}}_2\mathbf{n}_{1o}[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi = 0,\\
&\int_H \tilde{\boldsymbol{\sigma}}_2\mathbf{n}_{1o}\cdot\mathbf{n}[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi = 0.
\end{aligned} \tag{48}$$

In order to attain the Euler–Lagrange equations associated to Problem 6 the Green formula is recurrently employed. In this manner, in the distributional sense, it is straightforward to see that the Euler–Lagrange equations corresponding to the variational formulation (45) are the following

$$\begin{cases}
-\operatorname{div}_{\mathbf{x}_o} \mathbf{N}_{1t}^{H} - (\nabla_{\mathbf{x}_o}\mathbf{n})\mathbf{Q}_{1t}^{H} = \mathbf{f}_t^H & \text{in } \Sigma_o, \\
-\operatorname{div}_{\mathbf{x}_o} \mathbf{M}_{1t}^{H} + \mathbf{Q}_{1t}^{H} = \mathbf{m}_t^H & \text{in } \Sigma_o, \\
-\operatorname{div}_{\mathbf{x}_o} \mathbf{Q}_{1t}^{H} + \mathbf{N}_{1t}^{H} \cdot \nabla_{\mathbf{x}_o}\mathbf{n} = f_n^H & \text{in } \Sigma_o, \\
-\operatorname{div} \boldsymbol{\sigma}_2 = \mathbf{f} & \text{in } \Omega_2, \\
\mathbf{u}_{1t}^{o} = \bar{\mathbf{u}}_{1t}^{o} & \text{on } \partial\Sigma_{oD}, \\
\boldsymbol{\omega}_{1t} = \bar{\boldsymbol{\omega}}_{1t} & \text{on } \partial\Sigma_{oD}, \\
u_{1n} = \bar{u}_{1n} & \text{on } \partial\Sigma_{oD}, \\
\mathbf{u}_2 = \bar{\mathbf{u}}_2 & \text{on } \Gamma_{D2}, \\
\mathbf{N}_{1t}^{H}\mathbf{n}_{1o} = \bar{\mathbf{t}}_{1t}^{H} & \text{on } \partial\Sigma_{oN}, \\
\mathbf{M}_{1t}^{H}\mathbf{n}_{1o} = \bar{\mathbf{m}}_{1t}^{H} & \text{on } \partial\Sigma_{oN}, \\
\mathbf{Q}_{1t}^{H} \cdot \mathbf{n}_{1o} = \bar{t}_{1n}^{H} & \text{on } \partial\Sigma_{oN}, \\
\boldsymbol{\sigma}_2\mathbf{n}_2 = \bar{\mathbf{t}}_2 & \text{on } \Gamma_{N2}, \\
\gamma(\mathbf{u}_{1t}^{o} - \mathbf{u}_{2t}^{o}) = 0 & \text{on } \partial\Sigma_{oa}, \\
\gamma(\boldsymbol{\omega}_{1t} - \boldsymbol{\omega}_{2t}^{o}) = 0 & \text{on } \partial\Sigma_{oa}, \\
\gamma(u_{1n} - u_{2n}^{o}) = 0 & \text{on } \partial\Sigma_{oa}, \\
(1-\gamma)(\mathbf{u}_{1t}^{o} + \xi\boldsymbol{\omega}_{1t} + u_{1n}\mathbf{n} - \mathbf{u}_2) = 0 & \text{on } \Gamma_a, \\
\gamma\mathbf{t}_{1t}^{o} + (1-\gamma)\boldsymbol{\Pi}_t \int_H \mathbf{t}_2[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi = \mathbf{N}_{1t}^{H}\mathbf{n}_{1o} & \text{on } \partial\Sigma_{oa}, \\
\gamma\boldsymbol{\mu}_{1t} + (1-\gamma)\boldsymbol{\Pi}_t \int_H \mathbf{t}_2\xi[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi = \mathbf{M}_{1t}^{H}\mathbf{n}_{1o} & \text{on } \partial\Sigma_{oa}, \\
\gamma t_{1n} + (1-\gamma) \int_H \mathbf{t}_2\cdot\mathbf{n}[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi = \mathbf{Q}_{1t}^{H}\cdot\mathbf{n}_{1o} & \text{on } \partial\Sigma_{oa}, \\
\gamma\mathbf{t}_{1t}^{o} + (1-\gamma)\boldsymbol{\Pi}_t \int_H \mathbf{t}_2[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi = H_{\xi^0}\boldsymbol{\Pi}_t\bar{\boldsymbol{\sigma}}_2\mathbf{n}_{1o} & \text{on } \partial\Sigma_{oa}, \\
\gamma\boldsymbol{\mu}_{1t} + (1-\gamma)\boldsymbol{\Pi}_t \int_H \mathbf{t}_2\xi[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi = H_{\xi^1}\boldsymbol{\Pi}_t\bar{\boldsymbol{\sigma}}_2\mathbf{n}_{1o} & \\
\qquad\qquad + \boldsymbol{\Pi}_t \int_H \tilde{\boldsymbol{\sigma}}_2\mathbf{n}_{1o}\xi[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi & \text{on } \partial\Sigma_{oa}, \\
\gamma t_{1n} + (1-\gamma) \int_H \mathbf{t}_2\cdot\mathbf{n}[\boldsymbol{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi = H_{\xi^0}\bar{\boldsymbol{\sigma}}_2\mathbf{n}_{1o}\cdot\mathbf{n} & \text{on } \partial\Sigma_{oa}, \\
\Big[(1-\gamma)\mathbf{t}_2 - \tilde{\boldsymbol{\sigma}}_2\mathbf{n}_{1o}\Big] \perp \mathbf{v}_{2r} \quad \forall \mathbf{v}_{2r} \text{ satisfying (38)} & \text{on } \Gamma_a,
\end{cases} \tag{49}$$

where it has been assumed, without loss of generality, that over $\partial\Sigma_{oD}$ we have imposed Dirichlet conditions for the three fields involved in the shell model.

Here, the last eleven expressions above comprise the natural coupling conditions given by the variational formulation (45). The Euler–Lagrange equations put in evidence that the equivalence with respect to the real parameter $\gamma$ is actually lost, which is strongly related to that commented in remark 8. Hence, choosing $\gamma = 1$ yields to the following coupling conditions

$$
\begin{aligned}
\mathbf{u}_{1}{}_t^o &= \frac{1}{H_{\xi^0}}\mathbf{\Pi}_t \int_H \left(\mathbf{u}_2 - \frac{H_{\xi^1}}{H_{\xi^0}}\frac{\partial \mathbf{u}_2}{\partial \xi}\right)[\mathbf{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi && \text{on } \partial\Sigma_{oa}, && (50)\\
\boldsymbol{\omega}_{1t} &= \frac{1}{H_{\xi^0}}\mathbf{\Pi}_t \int_H \frac{\partial \mathbf{u}_2}{\partial \xi}[\mathbf{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi && \text{on } \partial\Sigma_{oa}, && (51)\\
u_{1n} &= \frac{1}{H_{\xi^0}}\int_H \mathbf{u}_2\cdot\mathbf{n}[\mathbf{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi && \text{on } \partial\Sigma_{oa}, && (52)\\
\mathbf{N}_{1}{}_t^H\mathbf{n}_{1o} &= H_{\xi^0}\mathbf{\Pi}_t\boldsymbol{\sigma}_2\mathbf{n}_{1o} && \text{on } \Gamma_a, && (53)\\
\mathbf{M}_{1}{}_t^H\mathbf{n}_{1o} &= H_{\xi^1}\mathbf{\Pi}_t\boldsymbol{\sigma}_2\mathbf{n}_{1o} && \text{on } \Gamma_a, && (54)\\
\mathbf{Q}_{1}{}_t^H\cdot\mathbf{n}_{1o} &= H_{\xi^0}\boldsymbol{\sigma}_2\mathbf{n}_{1o}\cdot\mathbf{n} && \text{on } \Gamma_a, && (55)\\
\mathbf{t}_{1}{}_t^o &= \mathbf{N}_{1}{}_t^H\mathbf{n}_{1o} && \text{on } \partial\Sigma_{oa}, && (56)\\
\boldsymbol{\mu}_{1t} &= \mathbf{M}_{1}{}_t^H\mathbf{n}_{1o} && \text{on } \partial\Sigma_{oa}, && (57)\\
t_{1n} &= \mathbf{Q}_{1}{}_t^H\cdot\mathbf{n}_{1o} && \text{on } \partial\Sigma_{oa}, && (58)
\end{aligned}
$$

where, the fact that

$$\tilde{\boldsymbol{\sigma}}_2\mathbf{n}_{1o} \perp \mathbf{v}_{2r} \quad \forall \mathbf{v}_{2r} \text{ satisfying (38)} \quad \text{on } \Gamma_a, \tag{59}$$

implies that $\tilde{\boldsymbol{\sigma}}_2\mathbf{n}_{1o}$ must be the null element, and therefore $\boldsymbol{\sigma}_2\mathbf{n}_{1o} = \bar{\boldsymbol{\sigma}}_2\mathbf{n}_{1o}$. This means that, on $\Gamma_a$, $\boldsymbol{\sigma}_2\mathbf{n}_{1o}$ is characterized just in terms of the generalized quantities $\mathbf{N}_{1}{}_t^o\mathbf{n}_{1o}$, $\mathbf{M}_{1}{}_t^o\mathbf{n}_{1o}$ and $\mathbf{Q}_{1}{}_t^H\cdot\mathbf{n}_{1o}$.

Reciprocally, taking $\gamma \neq 1$ leads to the following coupling conditions

$$
\begin{aligned}
(\mathbf{u}_{1}{}_t^o + \xi\boldsymbol{\omega}_{1t} + u_{1n}\mathbf{n}) &= \mathbf{u}_2 && \text{on } \Gamma_a, && (60)\\
\mathbf{N}_{1}{}_t^H\mathbf{n}_{1o} &= \mathbf{\Pi}_t\int_H \boldsymbol{\sigma}_2\mathbf{n}_{1o}[\mathbf{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi && \text{on } \partial\Sigma_{oa}, && (61)\\
\mathbf{M}_{1}{}_t^H\mathbf{n}_{1o} &= \mathbf{\Pi}_t\int_H \boldsymbol{\sigma}_2\mathbf{n}_{1o}\xi[\mathbf{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi && \text{on } \partial\Sigma_{oa}, && (62)\\
\mathbf{Q}_{1}{}_t^H\cdot\mathbf{n}_{1o} &= \int_H \boldsymbol{\sigma}_2\mathbf{n}_{1o}\cdot\mathbf{n}[\mathbf{\Lambda}^2\mathbf{m}_{1o}\cdot\mathbf{m}_{1o}]^{1/2}\,\mathrm{d}\xi && \text{on } \partial\Sigma_{oa}, && (63)\\
\mathbf{t}_2 &= \boldsymbol{\sigma}_2\mathbf{n}_{1o} && \text{on } \Gamma_a. && (64)
\end{aligned}
$$

Observe that expression (64) is obtained by combining the last four Euler–Lagrange equations seen in (49) and the decomposition given in (47).

From equations (50), (51) and (60) it becomes evident the need for the additional regularity for field $\mathbf{u}_2$ regardless of the value of $\gamma$. In particular, expressions (50), (51) and (52) state that

$$
\begin{aligned}
&\mathbf{u}_{2}{}_t^o = \mathbf{u}_{1}{}_t^o, && \text{on } \partial\Sigma_{oa}, && (65)\\
&\boldsymbol{\omega}_{2}{}_t^o = \boldsymbol{\omega}_{1t} && \text{on } \partial\Sigma_{oa}, && (66)\\
&u_{2}{}_n^o = u_{1n} && \text{on } \partial\Sigma_{oa}, && (67)\\
&\mathbf{u}_{2r} \text{ is arbitrary, but satisfying (38)} && \text{on } \Gamma_a, && (68)
\end{aligned}
$$

while expression (60) implies that

$$\mathbf{u}_{2t}^{o} = \mathbf{u}_{1t}^{o}, \qquad \text{on } \partial\Sigma_{oa}, \tag{69}$$

$$\boldsymbol{\omega}_{2t}^{o} = \boldsymbol{\omega}_{1t} \qquad \text{on } \partial\Sigma_{oa}, \tag{70}$$

$$u_{2n}^{o} = u_{1n} \qquad \text{on } \partial\Sigma_{oa}, \tag{71}$$

$$\mathbf{u}_{2r} = 0 \qquad \text{on } \Gamma_a. \tag{72}$$

**Remark 9** *Although the coupling conditions appear to be very similar, they have a deeper and valuable meaning. In fact, the combination of* (50), (51) *and* (52) *(situation* $\gamma = 1$*) establishes that the displacement field is discontinuous, since no conditions are imposed over the fluctuation* $\mathbf{u}_{2r}$*, and nothing impedes it from assuming an arbitrary value always satisfying* (38) *as indicated by* (68)*. On the contrary, expression* (60) *(situation* $\gamma \neq 1$*) determines that the displacement field is actually continuous since it is strictly imposed that* $\mathbf{u}_{2r} = 0$ *according to* (72)*. Reasoning in the same way, but focusing the attention in the dual variables it can be seen that, for* $\gamma = 1$*, the set of expressions* (53)*,* (54) *and* (55) *entails the continuity of the traction, while the traction is discontinuous for* $\gamma \neq 1$ *as pointed out by* (61)*,* (62) *and* (63)*. This is because of the existence of a component of the traction, that is* $\tilde{\boldsymbol{\sigma}}_2\mathbf{n}_{1o}$*, orthogonal to fluctuations* $\mathbf{v}_{2r}$ *in the sense of the dual product. Hence, nothing impedes this component of the stress from taking an arbitrary value that makes the traction be discontinuous.*

The problem stated by formulation (45) is in fact a coupled problem, where coupling conditions are naturally derived from the formulation. In order to gain insight in the resulting formulations, when choosing different values of $\gamma$ we may think of splitting the coupled problem into two sub–problems. This interpretation can be regarded in the context of an iterative scheme where the solution of the coupled problem is obtained by performing iterations between both sub–problems. For instance, the first problem could involve the 3D problem, while the second one would correspond to the 2D shell problem. Let us see that both are properly fed with conditions over the coupling boundary $\Gamma_a$ provided by the other problem. Indeed, it can be easily noticed that there exist opposite situations for different values of $\gamma$. Let us focus the analysis on the sub–problem over $\Omega_1$. Hence, for $\gamma = 1$ the shell is provided with a Dirichlet boundary condition over the coupling interface $\Gamma_a$ depending upon quantities defined over the 3D solid domain (see (50)–(51)–(52)), whereas it feeds back the 3D solid domain with a Neumann boundary condition over $\Gamma_a$ (see (53)–(54)–(55)). Contrarily, for $\gamma \neq 1$ the shell is supplied with a Neumann boundary condition over $\Gamma_a$ coming from the stress state of the 3D solid domain (see (61)–(62)–(63)), whilst it feeds back the 3D solid domain with a Dirichlet boundary condition over the same boundary (see (60)). Table 1 below summaries the state of the quantities in the problem that result continuous and/or discontinuous across the coupling interface $\Gamma_a$ according to the value of $\gamma$.

| *Value of γ* | *Segregated problem in $\Omega_1$* | *Segregated problem in $\Omega_2$* | *Continuous quantity* | *Discontinuous quantity* |
|---|---|---|---|---|
| {1} | Dirichlet | Neumann | Traction (see (53)–(54)–(55)) | Displacement (see (50)–(51)–(52)) |
| [0,1) | Neumann | Dirichlet | Displacement (see (60)) | Traction (see (61)–(62)–(63)) |

Table 1
State of the quantities in the 3D–2D coupled problem for different values of $\gamma$.

## 4 Coupling 3D solid and 1D beam models

The problem of coupling a full 3D solid model and a 1D beam model does not comprise any further complications than those viewed in the previous section. Here the analysis is limited to the simple case of a straight beam that deflects on the $y$–$z$ plane as shown in Figure 3. More complex cases such as curved beams in the space can be treated using the same ideas. Once again, there exists an artificial internal boundary $\Gamma_a$ that splits the domain $\Omega$ into sub–domains $\Omega_1$ and $\Omega_2$. The particular form of the component and of the loadings acting on it allows us to reduce the 3D solid model over $\Omega_1$ to a 1D beam model, while over $\Omega_2$ the full kinematics is maintained. Hence, a discontinuity in the fields is introduced in the problem as a result of the incompatibility between the kinematics involved in the description of the whole component. In what follows we accelerate the presentation since this case follows from applying exactly the same ideas than those employed in the preceding section.

### *4.1 Kinematical assumptions*

The model considered here for the domain $\Omega_1$ is the Bernoulli beam, whose kinematics is expressed as

$$\mathbf{u}_1(\mathbf{x}) = u_{1y}(z)\mathbf{e}_y - y\frac{\partial u_{1y}}{\partial z}(z)\mathbf{e}_z, \tag{73}$$

implying that transversal sections remain orthogonal after the deformation, and that normal fibers (in the $\mathbf{e}_y$ direction) do not change their size. Over $\Omega_2$ the full kinematical description is still considered . Without loss of generality the boundary at $\{z_a\}$ is regarded as a Dirichlet boundary, while the lateral boundary of $\Omega_1$ is a Neumann boundary so as to reduce the model. For simplicity in illustrating the ideas, the internal boundary $\Gamma_a$ is normal to the axis of the beam such that $\mathbf{n}_1 \equiv \mathbf{e}_z$.

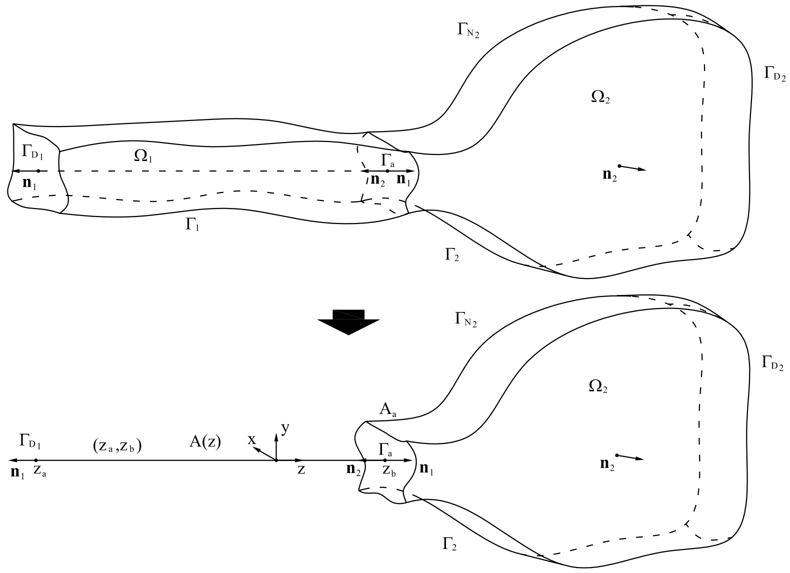


Fig. 3. Coupling 3D solid–1D beam models.

Going ahead as in the preceding section, once the kinematics over $\Omega_1$ is defined it is possible to give the explicit form of the duality product involved in the extended governing variational principle. Here, in accordance with expression (73) the displacement field is given by the scalar field $u_{1y} \in \boldsymbol{\mathcal{V}}_1$ where

$$\boldsymbol{\mathcal{V}}_1 = H^2((z_a, z_b)). \tag{74}$$

Once again, using duality arguments it is possible to give the explicit form of the duality product between admissible tractions $\mathbf{s}_1$ and displacement fields of the form of (73) as follows

$$\langle \mathbf{s}_1, \mathbf{u}_1 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = \left\langle (s_{1y}, \nu_{1x}), \left( u_{1y}, \frac{\partial u_{1y}}{\partial z} \right) \right\rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = (s_{1y} u_{1y})\Big|_{z_b} + \left( \nu_{1x} \frac{\partial u_{1y}}{\partial z} \right)\Big|_{z_b}, \tag{75}$$

where $z_b$ denotes the position of the boundary $\Gamma_a$. Here, the element $\mathbf{s}_1 = (s_{1y}, \nu_{1x})$ is the admissible traction in compliance with expression (73), being $s_{1y}$ and $\nu_{1x}$ the dual elements of $u_{1y}$ and $\frac{\partial u_{1y}}{\partial z}$ respectively, recalling that they are constants on $z_b$. Thus, we have

$$\begin{aligned} \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1) &= \mathbb{R} \times \mathbb{R}, \\ \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* &= \mathbb{R} \times \mathbb{R}. \end{aligned} \tag{76}$$

Again we face the problem of evaluating a term of the form $\langle \mathbf{s}_1, \mathbf{u}_2 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)}$, where $\mathbf{u}_2$ is an arbitrary function. Then, we need $\boldsymbol{\mathcal{U}}_2$ to be such that a decomposition of $\mathbf{u}_2$ according to expression (24) can be performed, and in such a case we have that $\mathbf{u}_{21}$ is

$$\mathbf{u}_{21}(\mathbf{x}) = u_{2y}^{o}\mathbf{e}_y - y\frac{\partial u_{2y}^{o}}{\partial z}\mathbf{e}_z \qquad \forall \mathbf{x} \in \Gamma_a, \tag{77}$$

being $u_{2y}^{o}$ and $\frac{\partial u_{2y}^{o}}{\partial z}$ constants. Accordingly, we have that

$$\langle \mathbf{s}_1, \mathbf{u}_2 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = \left\langle (s_{1y}, \nu_{1x}), \left( u_{2y}^{o}, \frac{\partial u_{2y}^{o}}{\partial z} \right) \right\rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = (s_{1y} u_{2y}^{o})\Big|_{z_b} + \left( \nu_{1x} \frac{\partial u_{2y}^{o}}{\partial z} \right)\Bigg|_{z_b}, \tag{78}$$

since the fluctuation component $\mathbf{u}_{2r}$ is such that

$$\langle \mathbf{s}_1, \mathbf{u}_{2r} \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)} = 0 \qquad \forall \mathbf{s}_1 \in \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^*, \tag{79}$$

in the same way as occurred in expression (37). Also, it is possible to point out the additional regularity here claimed for set $\boldsymbol{\mathcal{U}}_2$ as the counterpart statement to that made in remark 5. Thus, from (79) it is easy to see that the fluctuation in this case satisfies

$$\begin{aligned} &\int_{\Gamma_a} \mathbf{u}_{2r} \cdot \mathbf{e}_y \, \mathrm{d}\Gamma = 0, \\ &\int_{\Gamma_a} \mathbf{u}_{2r} \cdot \mathbf{e}_z \, \mathrm{d}\Gamma = 0. \end{aligned} \tag{80}$$

With this properties at hand we can characterize $u_{2y}$ and $\frac{\partial u_{2y}^{o}}{\partial z}$ as shown below

$$\begin{aligned} u_{2y}^{o} &= \frac{1}{A_a} \int_{\Gamma_a} \mathbf{u}_2 \cdot \mathbf{e}_y \, \mathrm{d}\Gamma, \\ \frac{\partial u_{2y}^{o}}{\partial z} &= \frac{1}{I_y^-} \int_{\Gamma_a} \mathbf{u}_2 \cdot \mathbf{e}_z \, \mathrm{d}\Gamma, \end{aligned} \tag{81}$$

where $I_y^- = \int_{\Gamma_a} (-y) \, \mathrm{d}\Gamma$ and $A_a = |\Gamma_a|$. After introducing all these elements we are able to define the regularity requirements of functions in $\Omega_2$

$$\boldsymbol{\mathcal{V}}_2 = \Bigg\{ \mathbf{v}_2 \in \mathbf{H}^1(\Omega_2);\ \mathbf{v}_{2|\Gamma_a} = v_{2y}^{o}\mathbf{e}_y - y\frac{\partial v_{2y}^{o}}{\partial z}\mathbf{e}_z + \mathbf{v}_{2r};\ \left( v_{2y}^{o}, \frac{\partial v_{2y}^{o}}{\partial z} \right) \in \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1);\ \mathbf{v}_{2r} \text{ satisfies } (80) \Bigg\}. \tag{82}$$

Analogously, it is

$$\langle \mathbf{s}_2, \mathbf{u}_2 \rangle_{\mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)^* \times \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_2)} = \int_{\Gamma_a} \mathbf{s}_2 \cdot \mathbf{u}_2 \, \mathrm{d}\Gamma. \tag{83}$$

With the purpose of reducing the full model, the integrals in $\Omega_1$ and $\Gamma_{N1}$ are written as follows

$$
\begin{aligned}
\int_{\Omega_1} (\cdot)\,\mathrm{d}\mathbf{x} &= \int_{z_a}^{z_b} \int_{\Gamma} (\cdot)\,\mathrm{d}\Gamma\,\mathrm{d}z, \\
\int_{\Gamma_{N_1}} (\cdot)\,\mathrm{d}\Gamma &= \int_{z_a}^{z_b} \int_{\partial\Gamma} (\cdot)\,\mathrm{d}\partial\Gamma\,\mathrm{d}z.
\end{aligned}
\tag{84}
$$

In this way, introducing the hypothesis (73) together with (75), (78) and (83) into the variational formulation (5), and considering the following generalized stress and loadings

$$
\begin{aligned}
M_1^{\Gamma} &= \int_{\Gamma} [\boldsymbol{\sigma}_1 \cdot (\mathbf{e}_z \otimes \mathbf{e}_z)](-y)\,\mathrm{d}\Gamma, \\
f_y^{\Gamma} &= \int_{\Gamma} \mathbf{f} \cdot \mathbf{e}_y\,\mathrm{d}\Gamma, \\
\bar{t}_{1y}^{\partial\Gamma} &= \int_{\partial\Gamma} \bar{\mathbf{t}}_1 \cdot \mathbf{e}_y\,\mathrm{d}\partial\Gamma,
\end{aligned}
\tag{85}
$$

it yields to the following problem that handles the coupling of a full 3D solid model and a 1D beam model that deflects in the $y$–$z$ plane:

**Problem 10** *For some $\gamma \in [0,1]$ find $((u_{1y}, \mathbf{u}_2), (t_{1y}, \mu_{1x}), \mathbf{t}_2) \in \boldsymbol{\mathcal{U}}_d \times \boldsymbol{\mathcal{Z}}_1 \times \boldsymbol{\mathcal{Z}}_2$ such that*

$$
\begin{aligned}
&\int_{z_a}^{z_b} M_1^{\Gamma} \frac{\partial^2 v_{1y}}{\partial z^2}\,\mathrm{d}z + \int_{\Omega_2} \boldsymbol{\sigma}_2 \cdot \nabla \mathbf{v}_2\,\mathrm{d}\mathbf{x} = \\
&\qquad \left(\gamma t_{1y} + (1-\gamma)\int_{\Gamma_a} \mathbf{t}_2 \cdot \mathbf{e}_y\,\mathrm{d}\Gamma\right) (v_{1y} - v_{2y}^{o})\Big|_{\Gamma_a} \\
&\qquad + \left(\gamma \mu_{1x} + (1-\gamma)\int_{\Gamma_a} [\mathbf{t}_2 \cdot \mathbf{e}_z](-y)\,\mathrm{d}\Gamma\right) \left(\frac{\partial v_{1y}}{\partial z} - \frac{\partial v_{2y}^{o}}{\partial z}\right)\Bigg|_{\Gamma_a} \\
&\quad - (1-\gamma)\int_{\Gamma_a} \mathbf{t}_2 \cdot \mathbf{v}_{2r}\,\mathrm{d}\Gamma + \gamma s_{1y}(u_{1y} - u_{2y}^{o})\Big|_{\Gamma_a} + \gamma \nu_{1x} \left(\frac{\partial u_{1y}}{\partial z} - \frac{\partial u_{2y}^{o}}{\partial z}\right)\Bigg|_{\Gamma_a} \\
&\qquad + (1-\gamma)\int_{\Gamma_a} \mathbf{s}_2 \cdot \left(u_{1y}\mathbf{e}_y - y\frac{\partial u_{1y}}{\partial z}\mathbf{e}_z - \mathbf{u}_2\right)\mathrm{d}\Gamma \\
&\qquad + \int_{z_a}^{z_b} f_y^{\Gamma} v_{1y}\,\mathrm{d}z + \int_{\Omega_2} \mathbf{f} \cdot \mathbf{v}_2\,\mathrm{d}\mathbf{x} + \int_{z_a}^{z_b} \bar{t}_{1y}^{\partial\Gamma} v_{1y}\,\mathrm{d}z + \int_{\Gamma_{N_2}} \bar{\mathbf{t}}_2 \cdot \mathbf{v}_2\,\mathrm{d}\Gamma \\
&\qquad\qquad \forall((v_{1y}, \mathbf{v}_2), (s_{1y}, \nu_{1x}), \mathbf{s}_2) \in \boldsymbol{\mathcal{V}}_d \times \boldsymbol{\mathcal{Z}}_1 \times \boldsymbol{\mathcal{Z}}_2,
\end{aligned}
\tag{86}
$$

*where $\boldsymbol{\mathcal{U}}_d = \boldsymbol{\mathcal{U}}_1 \times \boldsymbol{\mathcal{U}}_2$ with*

$$
\begin{aligned}
\boldsymbol{\mathcal{U}}_1 &= \left\{ u_{1y} \in \boldsymbol{\mathcal{V}}_1;\; u_{1y|z_a} = \bar{u}_{1y};\; \frac{\partial u_{1y}}{\partial z}\Big|_{z_a} = \bar{\alpha}_{1y} \right\}, \\
\boldsymbol{\mathcal{U}}_2 &= \{ \mathbf{v}_2 \in \boldsymbol{\mathcal{V}}_2;\; \mathbf{v}_{2|\Gamma_{D2}} = \bar{\mathbf{v}}_2 \},
\end{aligned}
\tag{87}
$$

*being $\boldsymbol{\mathcal{V}}_1$ and $\boldsymbol{\mathcal{V}}_2$ given as in* (74) *and* (82) *respectively. Here, $\boldsymbol{\mathcal{V}}_d$ is the associated linear space obtained from difference of elements of $\boldsymbol{\mathcal{U}}_d$. Also, $\boldsymbol{\mathcal{Z}}_1 = \mathcal{T}_{\Gamma_a}(\boldsymbol{\mathcal{V}}_1)^*$ as claimed in* (76)*, while $\boldsymbol{\mathcal{Z}}_2 = \mathbf{H}^{-1/2}(\Gamma_a)$. All the other elements are defined as in Problem 2.*

In order to close the problem proper constitutive laws relating, on one hand, the generalized stress $M_1^\Gamma$ with $u_{1y}$ and, on the other hand, the stress $\boldsymbol{\sigma}_2$ with $\mathbf{u}_2$ must be given.

**Remark 11** *Analogously to the statement of remark 7, Problem 10 demands more regularity for functions in $\Omega_2$. Again, this is a consequence of the kinematics chosen over $\Omega_1$, given by expression* (73)*. In view of that we notice, once again, that the assumptions taken over a portion of $\Omega$ affect the problem over the complementary portion of the domain.*

### *4.3 Euler–Lagrange equations*

Before looking for the Euler–Lagrange equations it is convenient to use a decomposition of the stress tensor similar to that shown in (47). In this case such a decomposition takes the following form

$$\boldsymbol{\sigma}_{2|\Gamma_a} = \bar{\sigma}_{2yz}(\mathbf{e}_y \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_y) + \bar{\sigma}_{2zz}(\mathbf{e}_z \otimes \mathbf{e}_z) + \tilde{\boldsymbol{\sigma}}_2, \tag{88}$$

where, in this case $\bar{\sigma}_{2yz}$, $\bar{\sigma}_{2zz}$ and $\tilde{\boldsymbol{\sigma}}_2$ are such that

$$\begin{aligned}
&\int_{\Gamma_a} \boldsymbol{\sigma}_2 \mathbf{e}_z \cdot \mathbf{e}_y \, \mathrm{d}\Gamma = A_a \bar{\sigma}_{2yz}, \\
&\int_{\Gamma_a} \boldsymbol{\sigma}_2 \mathbf{e}_z \cdot \mathbf{e}_z \, \mathrm{d}\Gamma = A_a \bar{\sigma}_{2zz}, \\
&\int_{\Gamma_a} \tilde{\boldsymbol{\sigma}}_2 \mathbf{e}_z \cdot \mathbf{e}_y \, \mathrm{d}\Gamma = 0, \\
&\int_{\Gamma_a} \tilde{\boldsymbol{\sigma}}_2 \mathbf{e}_z \cdot \mathbf{e}_z \, \mathrm{d}\Gamma = 0.
\end{aligned} \tag{89}$$

Hence, by using the Green formula we have, in the distributional sense, the following Euler–Lagrange equations corresponding to the variational formulation

(86)

$$
\begin{cases}
\dfrac{\partial^2 M_1^\Gamma}{\partial z^2} = f_y^\Gamma + \bar{t}_{1y}^{\partial\Gamma} & \text{in } (z_a, z_b),\\
-\operatorname{div}\boldsymbol{\sigma}_2 = \mathbf{f} & \text{in } \Omega_2,\\
u_{1y} = \bar{u}_{1y} & \text{in } \{z_a\},\\
\dfrac{\partial u_{1y}}{\partial z} = \bar{\alpha}_{1y} & \text{in } \{z_a\},\\
\mathbf{u}_2 = \bar{\mathbf{u}}_2 & \text{on } \Gamma_{D2},\\
\boldsymbol{\sigma}_2\mathbf{n}_2 = \bar{\mathbf{t}}_2 & \text{on } \Gamma_{N2},\\
\gamma(u_{1y} - u_{2y}^o) = 0 & \text{in } \{z_b\},\\
\gamma\left(\dfrac{\partial u_{1y}}{\partial z} - \dfrac{\partial u_{2y}^o}{\partial z}\right) = 0 & \text{in } \{z_b\},\\
(1-\gamma)\left(u_{1y}\mathbf{e}_y - y\dfrac{\partial u_{1y}}{\partial z}\mathbf{e}_z - \mathbf{u}_2\right) = 0 & \text{on } \Gamma_a,\\
\gamma t_{1y} + (1-\gamma)\displaystyle\int_{\Gamma_a} \mathbf{t}_2\cdot\mathbf{e}_y \, d\Gamma = -\dfrac{\partial M_1^\Gamma}{\partial z} & \text{in } \{z_b\},\\
\gamma\mu_{1x} + (1-\gamma)\displaystyle\int_{\Gamma_a} [\mathbf{t}_2\cdot\mathbf{e}_z](-y)\, d\Gamma = M_1^\Gamma & \text{in } \{z_b\},\\
\gamma t_{1y} + (1-\gamma)\displaystyle\int_{\Gamma_a} \mathbf{t}_2\cdot\mathbf{e}_y \, d\Gamma = A_a\bar{\sigma}_{2yz} & \text{in } \{z_b\},\\
\gamma\mu_{1x} + (1-\gamma)\displaystyle\int_{\Gamma_a} [\mathbf{t}_2\cdot\mathbf{e}_z](-y)\, d\Gamma = I_y^-\bar{\sigma}_{2zz} & \\
\qquad\qquad + \displaystyle\int_{\Gamma_a} \tilde{\boldsymbol{\sigma}}_2\mathbf{e}_z\cdot\mathbf{e}_z(-y)\, d\Gamma & \text{in } \{z_b\},\\
\Big[(1-\gamma)\mathbf{t}_2 - \tilde{\boldsymbol{\sigma}}_2\mathbf{e}_z\Big] \perp \mathbf{v}_{2r} \quad \forall \mathbf{v}_{2r} \text{ satisfying (80)} & \text{on } \Gamma_a.
\end{cases}
\tag{90}
$$

In this case the last eight expressions embody the coupling conditions obtained from the variational formulation (86). Once more, the equivalence concerning the real parameter $\gamma$ is lost as a result of the kinematical restrictions imposed. Indeed, choosing $\gamma = 1$ yields to the following set of coupling conditions

$$u_{1y} = \frac{1}{A_a}\int_{\Gamma_a} \mathbf{u}_2\cdot\mathbf{e}_y \, d\Gamma \qquad \text{in } \{z_b\}, \tag{91}$$

$$\frac{\partial u_{1y}}{\partial z} = \frac{1}{I_y^-}\int_{\Gamma_a} \mathbf{u}_2\cdot\mathbf{e}_z \, d\Gamma \qquad \text{in } \{z_b\}, \tag{92}$$

$$-\frac{\partial M_1^\Gamma}{\partial z} = A_a\boldsymbol{\sigma}_2\mathbf{e}_z\cdot\mathbf{e}_y \qquad \text{on } \Gamma_a, \tag{93}$$

$$M_1^\Gamma = I_y^-\boldsymbol{\sigma}_2\mathbf{e}_z\cdot\mathbf{e}_z \qquad \text{on } \Gamma_a, \tag{94}$$

$$t_{1y} = -\frac{\partial M_1^\Gamma}{\partial z} \qquad \text{in } \{z_b\}, \tag{95}$$

$$\mu_{1x} = M_1^\Gamma \qquad \text{in } \{z_b\}. \tag{96}$$

In reaching this result it has been used the fact that

$$\tilde{\boldsymbol{\sigma}}_2\mathbf{e}_z \perp \mathbf{v}_{2r} \quad \forall \mathbf{v}_{2r} \text{ satisfying (80)} \quad \text{on } \Gamma_a, \tag{97}$$

which, in turn, implies that $\tilde{\boldsymbol{\sigma}}_2\mathbf{e}_z$ is the null element, and thus $\boldsymbol{\sigma}_2\mathbf{e}_z = \bar{\sigma}_{2yz}\mathbf{e}_y + \bar{\sigma}_{2zz}\mathbf{e}_z$.

On the contrary, taking $\gamma \neq 1$ the resultant coupling conditions are

$$u_{1y}\mathbf{e}_y - y\frac{\partial u_{1y}}{\partial z}\mathbf{e}_z = \mathbf{u}_2 \qquad \text{on } \Gamma_a, \tag{98}$$

$$-\frac{\partial M_1^\Gamma}{\partial z} = \int_{\Gamma_a} \boldsymbol{\sigma}_2\mathbf{e}_z \cdot \mathbf{e}_y \,\mathrm{d}\Gamma \qquad \text{in } \{z_b\}, \tag{99}$$

$$M_1^\Gamma = \int_{\Gamma_a} \boldsymbol{\sigma}_2\mathbf{e}_z \cdot \mathbf{e}_z(-y) \,\mathrm{d}\Gamma \qquad \text{in } \{z_b\}, \tag{100}$$

$$\mathbf{t}_2 = \boldsymbol{\sigma}_2\mathbf{e}_z \qquad \text{on } \Gamma_a. \tag{101}$$

In this case, expression (101) is obtained by combining the last three expressions of (90) jointly with the decomposition (88).

**Remark 12** *For this problem, the choice $\gamma = 1$ establishes, through expressions* (91) *and* (92)*, that the displacement field is discontinuous, as $\mathbf{u}_{2r}$ is not restricted, excepting satisfying* (80)*. In the opposite situation for $\gamma \neq 1$, expression* (98) *stands for the continuity of the displacement field since it is $\mathbf{u}_{2r} = 0$. Analogously, it can be observed that, for $\gamma = 1$, expressions* (93) *and* (94) *imply the continuity of the traction, while for $\gamma \neq 1$ the traction is discontinuous as indicated by* (99) *and* (100)*. As aforesaid in the case of coupling 3D solid–2D shell models, this is because $\tilde{\boldsymbol{\sigma}}_2\mathbf{e}_z$ is orthogonal to functions of the form of $\mathbf{v}_{2r}$ in the sense of the dual product, and therefore this component of the traction may take an arbitrary value.*

In the same manner as done in the previous section, table 2 summarizes the two possible interpretations for different values of $\gamma$ under the idea of the segregation of the coupled problem into two sub–problems, presenting in each case the state of the quantities.

| *Value of $\gamma$* | *Segregated problem in $\Omega_1$* | *Segregated problem in $\Omega_2$* | *Continuous quantity* | *Discontinuous quantity* |
|---|---|---|---|---|
| {1} | Dirichlet | Neumann | Traction (see (93)–(94)) | Displacement (see (91)–(92)) |
| [0,1) | Neumann | Dirichlet | Displacement (see (98)) | Traction (see (99)–(100)) |

Table 2
State of the quantities in the 3D–1D coupled problem for different values of $\gamma$.

## 5 Conclusions

In this work a variational framework for coupling models with incompatible underlying kinematics was developed. It was then applied to practical situations embodying the coupling of a 3D solid model with reduced structural models. In this manner, it was possible to give proper variational principles for consistently stating the coupling of full 3D solid models with either 2D Naghdi shell models and 1D Bernoulli beam models. Essentially, the theory was based upon the sense in which the resulting duality products, after introducing the kinematical assumptions, are redefined in the governing formulation. In this regard, the manipulation of these terms by means of a real parameter allowed us to modify the way in which the continuity of the quantities was deemed within the variational principle. The framework exhibited in this paper is totally general and permits to work with any kinematical model and any kind of material. It is also worth emphasizing, according to what was avowed in this work, that any kinematical restriction over one portion of the domain directly affects the regularity requirements over the complementary domain, with all the implications that this matter entails.

We strongly believe that the way in which the problem was understood throughout this article is of fundamental significance in the context of theoretical mechanics, since it permits the clear understanding of the theory behind the coupling of models exhibiting different kinematics and, in particular, with different dimensionality. Also, we believe that this approach is very worthwhile with the aim of developing numerical coupling schemes for dealing with the numerical approximation of such general kinematically incompatible models.

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